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*THE CULTIVATION OF ENDAMOEBA HISTOLYTICA**

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There have been many claims of the successful cultivation of *Endamoeba histolytica*, the parasite of amoebic dysentery but these in general have proved to be based upon errors, especially in the identification of the amoeba cultivated. Consequently the prevailing attitude of protozoölogists and medical men is that this parasite is not cultivatable by any of our present-day laboratory methods. Cutler in 1918 reported the cultivation of *E. histolytica* but his work was not confirmed by protozoölogists who followed his technic. Cutler's evidence however, appears somewhat convincing although he failed to furnish a detailed account of his work.

In the present study we have succeeded in obtaining cultures of an amoeba from two cases (patients "K" and "E") of amoebiasis, neither of which showed any other intestinal protozoön. We will briefly describe our methods of cultivating this amoeba and submit evidence establishing its identity as *Endamoeba histolytica*.

Quite by chance *E. histolytica* was obtained in culture by one of us (W. C. B.) on February 18, 1924. The stool from patient "K," a missionary to Africa who had had an attack of amoebic dysentery 2 years previously, was examined for protozoa and tubes of L. E. S. (Locke egg-serum) medium were inoculated for the purpose of cultivating any flagellates that might be present. The stool contained a few amoebae, the nucleus of which suggested that of *E. histolytica*. Four days later the cultures on examination were negative for flagellates but contained a fairly large number of amoebae, possessing a nucleus like that of *E. histolytica*. Subcultures were then made and the amoebae were continued in cultures for over 8 months (October 25, 1924) and through 152 subcultures.

A second strain of *E. histolytica* was obtained in culture from patient "E" on July 2, 1924, and it was continued by subculturing for 3 months, to October 1, when it was accidentally lost.

Six different stools were obtained from patient "K" and *E. histolytica* was successfully cultivated from each one. On February 28th when the

second stool was obtained one of us (J. D.) joined in the work and was able to grow the amoeba in another medium composed of blood agar slants covered with Locke solution (8 parts) and human serum (1 part).

In the stools from patient "K" cysts were found that were typical of *E. histolytica* and some contained characteristic chromatoid bodies. In the stool submitted on March 28, 1924, the amoebae were numerous and many of them contained red blood cells.

Only two stools were obtained from patient "E" but in each *E. histolytica* was found containing ingested red corpuscles, while cysts were absent. Patient "E" had had a troublesome colitis for the past year but no diagnosis of amoebic dysentery had been made. In both this patient and patient "K" an ulcerative colitis was revealed on proctoscopic examination. Both patients subsequently received emetin treatment after which their infection disappeared.

A presumptive identification of the amoebae as *E. histolytica* from the stools of patient "K" was made on the grounds that the organism corresponded to *E. histolytica* as regards its locomotion. nutrition, morphology, cysts and its apparent pathogenicity for the host. A true identification of *E. histolytica* was made after we produced amoebic colitis and dysentery in 4 kittens, in one also an hepatic abscess. These kittens were inoculated per rectum with the diluted feces, containing the amoeba of patient "K." *E. histolytica* was recovered in cultures from each of these 4 kittens.

An identification of the amoeba from patient "E" as *E. histolytica* was made on the basis that it too corresponded to this organism as regards locomotion, nutrition, morphology, and apparent pathogenicity for its host. The production of amoebic dysentery in a kitten with typical amoebic ulcers in the colon, established the identification of this amoeba as *E. histolytica*.

For the cultivation of the "K" and "E" strains of *E. histolytica* three kinds of medium were used. The first employed and continued with best success was the L. E. S. medium which had been previously used for the cultivation of intestinal flagellates of man (Boeck, 1924).

This medium was made by preparing slants of whole egg by coagulation in an inspissator with subsequent sterilization in an autoclave and covering these slants with a sterile mixture of Locke's solution (8 parts) and human serum (1 part). The liquid portion of this medium was sterilized by passage through a Berkefeld filter.

The next most favorable medium used to cultivate *E. histolytica* was devised in July (by J. D.) and called the L. E. A. (Locke egg-albumin) medium. It was prepared by covering the egg slants with Locke's solution containing 1 per cent of crystallized egg albumin. It has the advantage over the L. E. S. medium of being more readily prepared since the albumin is usually more available than human serum.

As was stated above, blood agar (semi-solid) slants covered with the Locke-serum mixture was also used but it was not employed as a matter of routine as were the L. E. A. and L. E. S. media which were generally dependable and gave good results.

The initial pH of the culture media was from 7.2 to 7.8 to obtain the best growth of the amoebae. The amoebae persisted for 5 or 6 days as a rule in the cultures but appeared in maximum number after 48 hours. For this reason new subcultures were made on every other day.

The amoebae in all our cultures were to be found in the bacterial debris at the bottom of the culture tube and on the surface of the egg slant. These organisms ingested bacteria which were present in the culture and also red corpuscles when these were added to the medium. The amoeba under cultivation corresponded in size, locomotion, nutrition and morphology to *E. histolytica*.

We first proved the pathogenicity of the "K" strain of *E. histolytica* from our cultures by inoculating a total of 16 kittens per rectum with amoebae taken either directly from our cultures or indirectly from scrapings of intestinal mucosa of kittens already infected with cultured amoebae. Eleven of these kittens (69%) became infected and 9 showed large macroscopic ulcers of the colon. The first kitten that became infected was inoculated with cultured *E. histolytica* of the "K" strain on March 24, 1924, after it had been passed through 25 subcultures. The last 2 kittens (Nos. 120 and 124) were inoculated on July 16 and July 31, respectively, after the 85th and 93rd subculture and 5 months after the original culture was obtained. These kittens developed amoebic colitis and dysentery and in kitten No. 120 an hepatic abscess occurred. The prolonged cultivation of *E. histolytica* therefore had no apparent effect on its virulence.

Cultures of the "E" strain of *E. histolytica* were then inoculated into 2 kittens and both of these developed severe amoebic dysentery and colitis.

E. histolytica was recovered in culture from all of the kittens infected with cultured amoebae with the exception of one case which was too long post mortem at the time of autopsy.

The ulcers produced by the amoebae in the colon of the kittens varied considerably in size and shape but they were characteristic in their pathological details, possessing overhanging edges with often a small entrance to the ulcer which expanded considerably in the submucosa, undermining the mucosa above. Microscopically the ulcers contained a mass of cellular debris filling the lumen of the ulcer, numerous amoebae lying adjacent to the gradually necrosing tissue, and showed practically no infiltration with polymorphonuclear leucocytes. The lesions produced in kittens by the cultured *E. histolytica* did not differ from those caused by inoculation of *E. histolytica* in the feces of the patients and indicated that the cultured forms were similarly pathogenic.

Apparently excystation occurred once in our cultures but we failed to obtain it subsequently although several attempts were made.

Encystment occurred once in one of the cultures of the "E" strain and the cysts were about 12 microns in diameter, containing 1, 2 or 4 nuclei and some showed the characteristic rod-shaped chromatoid bodies which occur in cysts of *E. histolytica*. Encystment could not, however, be induced by following the methods described by Cutler (1919).

Summary.—An amoeba has been obtained in culture from the feces of 2 patients "K" and "E" which agrees with *E. histolytica* as regards its locomotion, nutrition morphology and pathogenicity necessitating its identification as this species.

Locke-egg-serum (L. E. S.) and the Locke-egg-albumin (L. E. A.) have proved satisfactory media for the cultivation of *E. histolytica*. These media support growth for 5 to 6 days although the maximum number of amoebae are found after 48 hours.

The "K" strain was kept through 152 subcultures by transferring every other day in L. E. S. medium from February 18 to October 25, 1925. The "E" strain was grown in culture from July 2 to October 1, 1925, and passed through 47 subcultures.

In culture *E. histolytica* feeds upon bacteria and red blood cells whenever the latter are furnished in the medium. The ingestion of bacteria by *E. histolytica* when deprived of red blood corpuscles or cell fragments and juices, shows that some of our previous views regarding the nutrition of *E. histolytica* have been erroneous and that under certain conditions this amoeba like *E. coli* and others may eat bacterial organisms.

The cultured *E. histolytica* proved to be as pathogenic for kittens as were the forms taken from the feces of the patients. They produced amoebic dysentery with ulcerative colitis and also in 1 case an hepatic abscess. Likewise the same type of lesions were produced in kittens infected directly by inoculation of the patient's feces containing *E. histolytica*.

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THE RAT AS A POSSIBLE CARRIER OF THE DYSENTERY AMOEBA

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The infection of rats by feeding them human feces containing the cysts of *Endamoeba histolytica* has been reported by (Lynch, 1915) who also claims to have observed this organism in a normal wild rat. The lack of detailed description and figures of the amoeba which he found in his rats leaves its identity uncertain. Later on Brug (1919) through similar experiments confirmed Lynch's observations. Kessel (1923) reported the transmission of various species of parasitic amoebae including *E. histolytica* from the human being to the rat. He at the same time made a study of the amoebae occurring spontaneously in the rat, describing the new species *E. ratti* and reclassifying previously described amoebae of the rat and mouse as *Councilmania muris* and *Councilmania decumani*. None of these investigators utilized the biological test of inoculating kittens with the amoeba in question but based their claims chiefly on morphological grounds.

In the present investigation the experimental infection of young rats that had been raised free from all parasitic amoebae was attempted. Infection having been obtained the morphological study of the organism was supplemented by the inoculation of young kittens in order to determine its transmissability and pathogenicity. A morphological study of the parasitic amoebae occurring spontaneously in white rats has revealed an endamoeba which if not *E. histolytica*, cannot as yet be differentiated with certainty from it. Another species resembling *Endolimax nana* of the human being has also been found in the laboratory rat. Throughout this study the method previously employed by Dr. Boeck and Dr. Drbohlav (1925)* for the cultivation of *E. histolytica* has been utilized.

Since the laboratory rat is commonly infected with several species of parasitic amoebae it is of primary importance in studying experimental infections to obtain stock which is amoeba-free. Kessel has advocated the employment of purgatives for detecting amoebic infection in rats but this method has certain disadvantages which are overcome in the high-enema method employed by the author in examining rats for the presence of amoebae. In order to obtain a clean stock of young rats free from amoebae, breeding rats are selected which by high enema examination appear to be free from infection. These are placed in sterilized cages, and fed with the exception of pasteurized milk, upon sterilized food. All sawdust, nesting material, etc., which is introduced into the cage is pre-

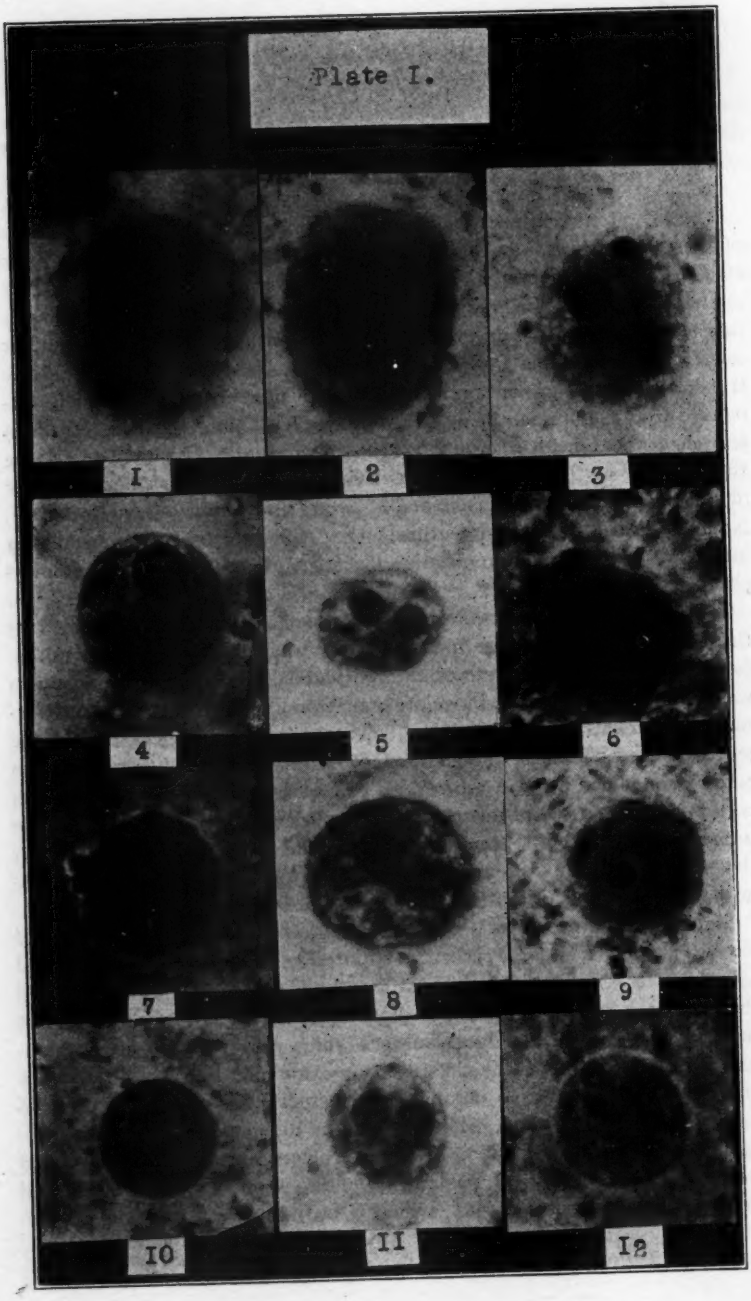
Plate I.
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PLATE I

All figures in this plate are photographed from preparations fixed with Schaudinn's solution, and stained with aqueous iron hematoxyline. Magnification 1750 diameters.

Figs. 1 to 6. E. histolytica from experimental rats and kittens.

Fig. 1. Original material from kitten used for the first series of experiment. (With red cells.)

Fig. 2. Trophozoite from infected rat.

Fig. 3. Trophozoite from kitten which had been infected with material from experimental rats.

Fig. 4. Uninucleate cyst from infected rat.

Fig. 5. Binucleate cyst from infected rat.

Fig. 6. Quadrinucleate cyst from infected rat.

Figs. 7 to 12. E. histolytica (murina) found in normal laboratory rats.

Fig. 7. Trophozoite from normal laboratory rats.

Fig. 8. Trophozoite in culture.

Fig. 9. Trophozoite from kitten infected with E. histolytica (murina). (Showing red blood cell.)

Fig. 10. Uninucleate cyst from normal laboratory rat.

Fig. 11. Binucleate cyst from normal laboratory rat.

Fig. 12. Quadrinucleate cyst from normal laboratory rat.

viously sterilized in the autoclave 20 minutes at 15 pounds pressure. The pregnant mother is isolated under such conditions and the young are removed to another sterilized cage at the age of 3 or 4 weeks. After attaining the age of six weeks each is subjected to at least two high-enema examinations before being used for experimental infection.

Experimental Transmission of E. Histolytica of Human Origin to the Rat.—Two series of experiments were carried out. Material for the inoculation of the first series, derived from a kitten infected with the *"E" strain of E. histolytica* was supplied through the courtesy of Dr. Boeck and Dr. Drbohlav. Two amoeba-free adult rats receiving a rectal injection of this material showed mobile amoeba in seven and ten days, respectively, after the injection. Fecal material obtained by high enema from one of these rats was injected rectally into two young kittens, both of which developed an amoebiasis with lesions of the large intestine. The study of stained section of the affected portions of intestinal wall showed amoebae in the tissues. Fecal material from one of these kittens was injected rectally into two young amoeba-free rats but only one of them became infected.

Material for the second series of experimental infections was furnished by Dr. S., he himself being a healthy carrier of *E. histolytica*, *E. coli* and *Endolimax nana*. The feces was fed on November 13, 15 and 17 to a litter of nine young amoeba-free rats. Neither *E. coli* or *Endolimax nana* became established in any of these rats but they showed instead an infection with an amoeba of the type of *E. histolytica*. In order to test the pathogenicity of this organism fecal material from these rats was injected into a young kitten. This became ill and showing dysentery with amoebae

in the feces on the third day after inoculation, was killed. Amoebae were found in the intestinal mucosa in stained sections. Three months after the infection of the rats, another kitten was injected with fecal material from them but died of pneumonia without developing amoebiasis. Four young amoeba-free rats subsequently placed wth the infected ones, became infected through exposure. Periodical observations on the rats infected with these two strains of *E. histolytica* have shown that the infection is at its height at from one to two months after its first appearance and thereafter tends to gradually diminish, but may persist in certain animals for as long as four months.

No macroscopic lesions were found in any of the rats killed during the period of infection and no amoebae were demonstrable in stained sections of the large intestine. Cysts with from one to four nuclei were discharged in the feces of the infected rats.

The results of these experiments indicate that *E. histolytica* is transmissible to the white rat either by rectal injection or by feeding and that the resulting infection continues over a period of several months. The infection is also transmitted from the infected rat to young amoeba-free rats by exposure in the same cage. The pathogenicity of the amoeba employed is demonstrated by transference to kittens in which it produces lesions with amoebae in the affected tissue.

Parasitic Amoebae Found Occurring Spontaneously in the Laboratory White Rat.—Eight nucleated cysts have been described for all species of amoeba thus far reported as occurring naturally in the rat and mouse. In the present investigation one species of endamoeba producing eight nucleated cysts was found in rats. Of greater interest was the occurrence of an endamoeba producing four nucleated cysts and an amoeba closely resembling *Endolimax nana* of the human being. The former since it has not been possible to definitely differentiate it from *E. histolytica* of the human being is designated *E. histolytica* (var. *murina*) and the latter on the ground of biological differences is designated as *Endolimax ratti*.

Endamoeba Histolytica (*Murina*).—In a rat killed from a lot obtained from another department an endamoeba was found which grew readily on the medium of Boeck and Drbohlav. On further examination, six other rats of this lot were found infected with the same amoeba. The study of this amoeba has revealed so few characters not wholly in agreement with *E. histolytica* of man that only a brief discussion of its morphology will be given at this time. The trophozoites when rounded measure from 12 to 20 microns. Its locomotion is characteristically slug-like as in the case of the human species. In preparations stained with Heidenhain's iron haematoxylin, the nucleus shows the karyosome centrally situated in about 60 per cent of the trophozoites, in the remainder being slightly

eccentric. Except for a less uniform central location of the karyosome, the nucleus agrees in every respect with that of the parasite in human dysentery. The inclusion of large numbers of bacteria in the cytoplasm is another possible difference although it must be taken into account that the trophozoites here studied are obtained directly from the colon of killed animals whereas those of the human being are usually studied from stools.

Cysts in the one, two and four nucleated stages are found in the feces of the rat and measure from 10 to 13 microns. These show chromatoid bodies of various sizes and shapes. In the one nucleated cysts these are identical with those of *E. histolytica* but the typical rod-shaped bars of this material have not been observed in the four nucleated cyst.

E. histolytica (*murina*) grows readily in the culture media employed by Boeck and Drbohlav and also on a modification of this medium prepared by substituting 100 cc. of beef broth in place of 25 cc. of the Locke Ringer solution in the preparation of the egg slant and omitting the serum in the fluid subsequently added. Encystment has occurred regularly on this medium coincident with concentration through evaporation.

The rectal injection of cultures of this amoeba into young amoeba-free rats produces infection without evidence of disease. The organism is also readily transmitted from infected rats to young normal ones by exposure.

Eight kittens have been injected with *E. histolytica* (*murina*). Five of these became infected, two temporarily, complete disappearance of the amoeba following. One of these kittens showed amoebae containing red corpuscles during life but no lesions were found at post mortem and no amoebae were demonstrable in the tissues in stained section. Another showed amoebae with red cells for a time but very few amoeba at post mortem. In one case there were macroscopic lesions with amoebae demonstrable in the tissue in stained sections.

Endolimax Ratti (*Spec. Nov.*).—Although Kessel reports experimental infection of white rats with *Endolimax nana*, this, as already noted, has failed in the present investigation. In all, fourteen rats were fed feces containing *Endolimax nana* cysts, from three different human cases, but none became infected. An amoeba of this type was, however, found occurring naturally in laboratory rats. The morphology of this organism is identical with that of *End. nana*, but the fact that it parasitizes the rat makes it appear to be at least biologically distinct. The tentative name *Endolimax ratti* is proposed for this amoeba. The mature cysts show four nuclei. The general characters of this organism are illustrated in figures 17 to 22.

Summary.—From the results of the present experiments, it is apparent that *E. histolytica* of human origin may be transmitted to white rats. This may be accomplished either by the rectal inoculation of material containing the trophozoites or by the feeding of material containing its cysts.

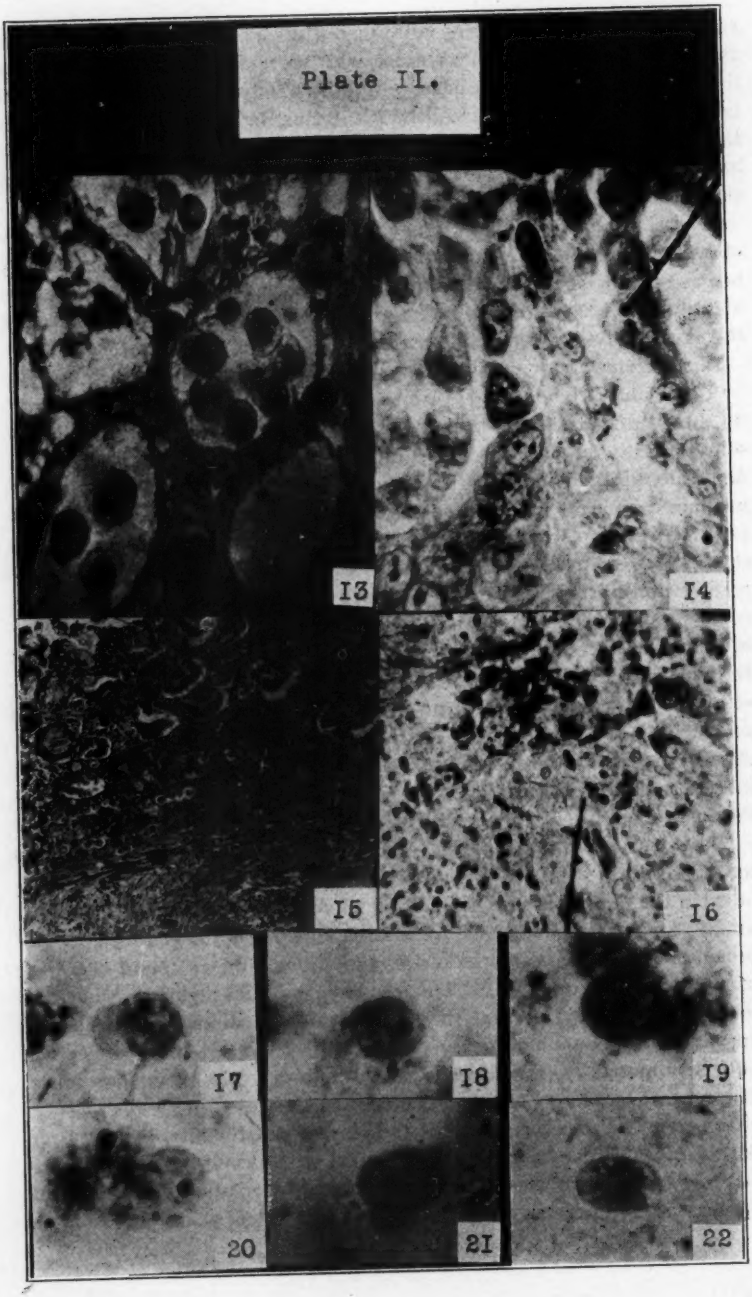
Plate II.
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PLATE II

Figures 13 to 16 in this plate are photographed from preparations fixed with sublimate alcohol (with acetic acid) and stained by Mann's methylblue-eosin method. No estimation on magnification.

Figures 17 to 22 in this plate are photographed from preparations fixed with Schaudinn's solution and stained with aqueous iron hematoxylin. Magnification 1750 diameters.

Fig. 13. Section of large intestine of kitten infected with E. histolytica (E. strain) from infected rat.

Fig. 14. Section of large intestine of kitten infected with E. histolytica (Murina) from normal laboratory rat.

Fig. 15. Section of large intestine of kitten infected with E. histolytica (E. strain) from infected rat.

Fig. 16. Section of large intestine of kitten infected with E. histolytica (S. strain) from infected rat.

Figures 17 to 22. Endolimax muris (sp. nov.) from normal laboratory rats.

Figures 17 to 20. Trophozoites with characteristic karyosome and pseudopodia.

Fig. 21. Precystic trophozoite.

Fig. 22. Four-nucleate cyst.

The pathogenicity of the amoeba found in the experimentally infected rat has been demonstrated by the inoculation of kittens with the production of dysentery and lesions containing amoebae.

The infection of rats with *E. histolytica* of human origin is readily transmitted from rat to rat by association in the same cage. The height of the infection occurs between one and two months after which the organisms tend to gradually diminish in number but may persist for at least four months.

An endamoeba here designated *E. histolytica* (var. *murina*) agreeing very closely in its morphology with *E. histolytica* of human origin is found occurring spontaneously in the laboratory rat. Like the human parasite it is very readily cultivated and is transmissible to kittens for which it appears to be only mildly pathogenic.

Encystment of this organism is readily produced in suitable medium.

An amoeba morphologically similar to *Endolimax nana* but possibly biologically distinct has been found in laboratory rats. This temporarily designated as *Endolimax ratti* requires further investigation to establish it as a distinct species.

The results of this investigation incriminate the rat as a possible carrier of E. histolytica, the causal agent of amoebic dysentery.

* See preceding paper in this issue.

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LINEAR CONNECTIONS OF A SPACE WHICH ARE DETERMINED BY SIMPLY TRANSITIVE CONTINUOUS GROUPS

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1. The concept of parallelism introduced by Levi-Civita for general Riemannian manifolds of n dimensions has been extended by many writers to manifolds for which there is not an assigned metric. Weyl[1] used the term *affine connection* to define the relation between elements at different points of a space and used a linear connection in the definition of which the coefficients Γ^i_{jk} are symmetric in j and k. Equivalent definitions have been used by Eddington,[2] and by Veblen[3] and the author[3] in their papers on the *geometry of paths*. Schouten[4] has made an analysis of types of linear connection without the restriction that the coefficients be symmetric. It is the purpose of this note to show that the infinitesimal generators of a simply transitive continuous group in n variables serve to define an unsymmetric linear connection for which there exist symmetric tensors g_{ij}, involving $n(n+1)/2$ arbitrary constants, whose first covariant derivatives are zero. Moreover, if one of these tensors is taken for the fundamental tensor of a Riemannian manifold, the group is a transitive group of motions for the manifold.

2. Consider a space of n dimensions of coördinates x^i for $i = 1, \ldots, n$. Let $\lambda^i_{\alpha/}$ be the components of n linearly independent contravariant vectors in the space; in this notation a $\lambda^i_{\alpha/}$ given α for $\alpha = 1, \ldots, n$ indicates the vector and i for $i = 1, \ldots, n$ the component. Since the vectors are linearly independent, their determinant $\Lambda = \left| \lambda^i_{\alpha/} \right|$ is different from zero. Let Λ^α_i be the cofactor of $\lambda^i_{\alpha/}$ in Λ divided by Λ; then

$$\lambda^i_{\alpha/}\, \Lambda^\beta_i = \epsilon^\beta_\alpha, \tag{2.1}$$

where i is summed from 1 to n, according to the customary convention which will be followed in this paper, and where ϵ^β_α is 1 for $\alpha = \beta$ and 0 for $\alpha \neq \beta$.

If we denote by $\lambda'^i_{\alpha/}$ the components of the vector $\lambda^i_{\alpha/}$ in a coördinate system x'^i, we have

$$\lambda^i_{\alpha/} = \lambda'^{\rho}_{\alpha/} \frac{\partial x^i}{\partial x'^{\rho}}. \tag{2.2}$$

From (2.1) and analogous equations for the $x''s$, we have

$$\lambda^i_{\alpha/} \Lambda^{\beta}_i = \lambda'^{\rho}_{\alpha/} \Lambda'^{\beta}_{\rho},$$

which are reducible by (2.2) to

$$\lambda'^{\rho}_{\alpha/} \left(\Lambda^{\beta}_i \frac{\partial x^i}{\partial x'^{\rho}} - \Lambda'^{\beta}_{\rho} \right) = 0$$

Since determinant $\Lambda' \neq 0$, we have

$$\Lambda'^{\beta}_{\rho} = \Lambda^{\beta}_i \frac{\partial x^i}{\partial x'^{\rho}}. \tag{2.3}$$

Hence for each value of β, the quantities Λ^{β}_i are the components of a covariant vector, i indicating the component.

3. From (2.3) we have by differentiation

$$\frac{\partial \Lambda'^{\beta}_{\rho}}{\partial x'^{q}} = \frac{\partial \Lambda^{\beta}_i}{\partial x^j} \frac{\partial x^j}{\partial x'^{q}} \frac{\partial x^i}{\partial x'^{\rho}} + \Lambda^{\beta}_i \frac{\partial^2 x^i}{\partial x'^{\rho} \partial x'^{q}}.$$

Multiplying by

$$\lambda'^{r}_{\beta/} = \lambda^k_{\beta/} \frac{\partial x'^{r}}{\partial x^k},$$

and summing for β, we get

$$\lambda'^{r}_{\beta/} \frac{\partial \Lambda'^{\beta}_{\rho}}{\partial x'^{q}} = \lambda^k_{\beta/} \frac{\partial \Lambda^{\beta}_i}{\partial x^j} \frac{\partial x^i}{\partial x'^{\rho}} \frac{\partial x^j}{\partial x'^{q}} \frac{\partial x'^{r}}{\partial x^k} + \frac{\partial x'^{r}}{\partial x^k} \frac{\partial^2 x^k}{\partial x'^{\rho} \partial x'^{q}}. \tag{3.1}$$

If we put

$$\bar{\Gamma}^k_{ij} = \lambda^k_{\beta/} \frac{\partial \Lambda^{\beta}_i}{\partial x^j}, \tag{3.2}$$

equations (3.1) become

$$\bar{\Gamma}'^{r}_{\rho q} = \bar{\Gamma}^k_{ij} \frac{\partial x^i}{\partial x'^{\rho}} \frac{\partial x^j}{\partial x'^{q}} \frac{\partial x'^{r}}{\partial x^k} + \frac{\partial x'^{r}}{\partial x^k} \frac{\partial^2 x^k}{\partial x'^{\rho} \partial x'^{q}}. \tag{3.3}$$

These are the conditions which the functions $\bar{\Gamma}^k_{ij}$ must satisfy in order that they determine a linear connection, as Schouten[4] has shown. This particular result is due to Weitzenböck.[5]

From the definition of Λ^{β}_i we have also

$$\lambda^k_{\beta/} \Lambda^{\beta}_i = \delta^k_i \tag{3.4}$$

where δ_k^i is 0 or 1 according as $i \neq k$ or $i = k$. From this equation and (3.2), we have also

$$\bar{\Gamma}_{ij}^k = -\Lambda_i^\beta \frac{\partial \lambda_{\beta/}^k}{\partial x^j}. \tag{3.5}$$

Multiplying (3.2) and (3.5) by Λ_k^α and $\lambda_{\alpha/}^i$, respectively, and summing for k and i in these respective cases, we have in consequence of (2.1)

$$\frac{\partial \Lambda_i^\alpha}{\partial x^j} = \Lambda_k^\alpha \bar{\Gamma}_{ij}^k, \tag{3.6}$$

and

$$\frac{\partial \lambda_{\alpha/}^k}{\partial x^j} = -\lambda_{\alpha/}^i \bar{\Gamma}_{ij}^k. \tag{3.7}$$

Since these are the conditions that the first covariant derivative of Λ_i^α and $\lambda_{\alpha/}^k$ with respect to the $\bar{\Gamma}$'s be zero, we say that for the linear connection defined by (3.2) each field of vectors $\lambda_{\alpha/}^k$ (and Λ_i^α) forms a *parallel field*.

4. If in place of (3.2) and (3.5) we put

$$\Gamma_{ij}^k = \lambda_{\beta/}^k \frac{\partial \Lambda_j^\beta}{\partial x^i} = -\Lambda_j^\beta \cdot \frac{\partial \lambda_{\beta/}^k}{\partial x^i}, \tag{4.1}$$

the Γ's satisfy equations of the form (3.3) and consequently define another linear connection. In this case we have in place of (3.6) and (3.7)

$$\frac{\partial \Lambda_j^\alpha}{\partial x^i} = \Lambda_k^\alpha \Gamma_{ij}^k \tag{4.2}$$

and

$$\frac{\partial \lambda_{\alpha 1}^k}{\partial x^i} = -\lambda_{\alpha 1}^j \Gamma_{ij}^k. \tag{4.3}$$

For this linear connection the vectors $\lambda_{\alpha/j}$ and Λ_k^α do not form parallel fields.

In order that there may exist a symmetric tensor g_{ij} whose first covariant derivative with respect to the Γ's be zero, we must have

$$\frac{\partial g_{ij}}{\partial x^k} - g_{lj} \Gamma_{ik}^l - g_{il} \Gamma_{jk}^l = 0 \tag{4.4}$$

The conditions of integrability of these equations are

$$g_{ih} B_{jkl}^h + g_{hj} B_{ikl}^h = 0, \tag{4.5}$$

where

$$B_{jkl}^h = \frac{\partial \Gamma_{jl}^h}{\partial x^k} - \frac{\partial \Gamma_{jk}^h}{\partial x^l} + \Gamma_{jl}^m \Gamma_{mk}^h - \Gamma_{jk}^m \Gamma_{ml}^h. \tag{4.6}$$

In order that equations (4.4) be completely integrable, that is that they admit a solution involving $n(n+1)/2$ arbitrary constants, it is necessary

and sufficient that all the functions B^h_{jkl} be zero. We show in the next section that these conditions are satisfied, when the vectors $\lambda^i_{\alpha/}$ determine the infinitesimal generators of a simply transitive continuous group in n variables.

5. If

$$X_\alpha f \equiv \lambda^i_{\alpha/} \frac{\partial f}{\partial x^i} \tag{5.1}$$

are the generators of a simply transitive group, it is necessary that

$$(X_\alpha, X_\beta) f \equiv \lambda^i_{\alpha/} \frac{\partial}{\partial x^i}\left(\lambda^j_{\beta/} \frac{\partial f}{\partial x^j}\right) - \lambda^j_{\beta/} \frac{\partial}{\partial x^j}\left(\lambda^i_{\alpha/} \frac{\partial f}{\partial x^i}\right) = C^\gamma_{\alpha\beta} X_\gamma f, \tag{5.2}$$

where the C's are constants. If these constants satisfy the equations

$$C^\sigma_{\alpha\beta} C^\delta_{\sigma\lambda} + C^\sigma_{\lambda\alpha} C^\delta_{\sigma\beta} + C^\sigma_{\beta\lambda} C^\delta_{\sigma\alpha} = 0 \quad (\alpha,\beta,\lambda,\delta,\sigma = 1,..,n) \tag{5.3}$$

the conditions are sufficient as well as necessary.[6]

From (5.2) we have

$$\lambda^k_{\alpha/} \frac{\partial \lambda^h_{\beta/}}{\partial x^k} - \lambda^k_{\beta/} \frac{\partial \lambda^h_{\alpha/}}{\partial x^k} = C^\gamma_{\alpha\beta} \lambda^h_{\gamma/}.$$

Multiplying this equations by $\Lambda^\alpha_i \Lambda^\beta_j$ and summing for α and β, we have in consequence of (3.4) and (4.1)

$$\Gamma^h_{ij} - \Gamma^h_{ji} = - C^\gamma_{\alpha\beta} \lambda^h_{\gamma/} \Lambda^\alpha_i \Lambda^\beta_j. \tag{5.4}$$

By means of these equations, equations (4.6) can be put in the form

$$B^h_{jkl} = \frac{\partial \Gamma^h_{lj}}{\partial x^k} - \frac{\partial \Gamma^h_{kj}}{\partial x^l} - C^\gamma_{\alpha\beta}\left[\frac{\partial}{\partial x^k}(\lambda^h_{\gamma/} \Lambda^\alpha_j \Lambda^\beta_l) - \frac{\partial}{\partial x^l}(\lambda^h_{\gamma/} \Lambda^\alpha_j \Lambda^\beta_k)\right] + \Gamma^m_{jl} \Gamma^h_{mk} - \Gamma^m_{jk} \Gamma^h_{ml}.$$

Substituting in the first two terms expressions for the Γ's given by the second of (4.1), the resulting expressions are reducible to zero by means of (4.2), (4.3), (5.3) and (5.4).

6. If g_{ij} are the components of the fundamental tensor of a Riemannian manifold V_n of n dimensions, and $\lambda^i_{\alpha/}$ for $i = 1,.., n$ are a set of functions satisfying the equations

$$\lambda^l_{\alpha/} \frac{\partial g_{ij}}{\partial x^l} + g_{ih} \frac{\partial \lambda^h_{\alpha/}}{\partial x^j} + g_{hj} \frac{\partial \lambda^h_{\alpha/}}{\partial x^i} = 0, \tag{6.1}$$

then, as Killing[7] has shown, $X_\alpha f = \lambda^i_{\alpha/} \frac{\partial f}{\partial x^i}$ is the generator of a one parameter group of motions of V_n into itself. Conversely, suppose that (5.1) are the generators of a simply transitive group. In order that this be a group of motions for a V_n, it is necessary that (6.1) for $\alpha = 1,\ldots, n$ hold. Multiplying (6.1) by Λ^α_k and summing for α, we get (4.4) in consequence

of (3.4) and (4.1) Hence equations (6.1) admit solutions involving $n(n+1)/2$ arbitrary constants, and consequently there are $\infty^{\frac{n(n+1)}{2}}$ Riemannian manifolds admitting a given simply transitive group as a group of motions, a result due to Bianchi.[8]

[1] Space, Time, Matter, p. 112.
[2] The Mathematical Theory of Relativity, Chapter 7.
[3] These PROCEEDINGS, **8** and **9**; *Trans. Amer. Soc.*, **25** and **26**; *Annals*, **24**.
[4] Der Ricci-Kalkül, pp. 64, 65.
[5] Invariantentheorie, pp. 318, 319.
[6] Lie, Vorlesungen über Continuerliche Gruppen, pp. 391, 396.
[7] Crelle, **109**, 1892, p. 121.
[8] Lezioni Sulla Teoria dei Gruppi Continue Finiti, p. 517.

NOTE ON A THEOREM BY H. KNESER

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In a recent number of the *Proceedings of the Royal Academy of Amsterdam*,[1] H. Kneser proves the following theorem:

I. *A necessary and sufficient condition that a connected n-dimensional manifold* M^n $(n > 1)$ *be separated into two and only two regions by every connected* $(n-1)$*-dimensional manifold* M^{n-1} *contained in* M^n *is that the first Betti number* P^1 *of* M^n *be equal to unity and that there be no even coefficients of torsion of the lowest order.*

Kneser's proof is combinatorial and presupposes that M^{n-1} always belongs to a cellular subdivision of M^n.

It is perhaps worth noticing that the meaning of this theorem becomes very transparent in the light of the simplified, modulo 2 theory of connectivity, in which no distinction is made between positively and negatively oriented cells. Let us say that an n-complex is *completely connected* if it is possible to pass from any n-cell of the complex to any other by a series of steps at each of which we go from an n-cell E^n to an n-cell F^n incident to the same $(n-1)$-cell as E^n. Then, in the language of the modulo 2 theory, we have the following basic theorem which is almost self-evident.

II. *Let* C^n *be any closed, irreducible n-complex and* C^{n-1} *any closed irreducible* $(n-1)$*-complex made up of cells of* C^n. *Then, if* C^{n-1} *is bounding, it decomposes the complex* C^n *into exactly two completely connected complexes* C_1^n *and* C_2^n, *but if* C^{n-1} *is non-bounding, it leaves* C^n *completely connected.* The complexes C_1^n and C_2^n which touch along C^{n-1} may also touch in certain cells of dimensionalities less than $n-1$.

For suppose an irreducible sub-complex C^{n-1} of C^n decomposes C^n into two or more completely connected complexes C_i^n. Evidently the boundary of each of the regions C_i^n is made up of cells of C^{n-1}. Therefore, since C^{n-1} is irreducible, it must be the complete boundary of each complex C_i^n. It follows that if C^{n-1} is non-bounding, it has to leave C^n completely connected. On the other hand, if the irreducible complex C^{n-1} bounds a complex C_1^n, it also bounds the complex $C_2^n = C^n - C_1^n$. It cannot bound more than twice, otherwise C^n would contain $C_1^n + C_2^n$ as a proper closed sub-complex, contrary to the assumption about the irreducibility of C^n. This proves the theorem. As a corollary we have at once:

III. *A necessary and sufficient condition that every closed irreducible* $(n-1)$*-complex* C^{n-1} *decompose the closed, irreducible n-complex* C^n *into exactly two completely connected complexes* C_1^n *and* C_2^n *is that the* $(n-1)$ *-th connectivity number* R^{n-1} *be equal to unity.*

The condition $R^{n-1} = 1$ is the essential thing. If the complex C^n happens to determine a manifold, then, by the well known duality theorem,

$$R^{n-1} = R^1,$$

where R^1 is the first connectivity number of the manifold. Moreover, since we have the simple general relation.

$$R^1 = P^1 + h,$$

where P^1 is the first Betti number of the manifold and h the number of even coefficients of torsion of lowest order, we may replace the condition $R^{n-1} = 1$ by Kneser's condition that P^1 be equal to unity and that there be no even coefficients of torsion of lowest order.

To prove Theorem I, we now have only to prove (1) that if C^n of Theorem II determines a manifold, the completely connected complexes C_1^n, and C_2^n can touch in points of C^{n-1} only, otherwise C^n would possess a singular point, and (2) that if the irreducible complex C^{n-1} fails to determine a manifold M^{n-1} because of singularities, there is always a neighboring irreducible complex C_0^{n-1} without singularities and homologous to C^{n-1}. This may be proved in the manner indicated by Kneser.[2]

[1] *Proc. Amsterdam*, **27**, Nos. 7 and 8, Sept. 27, 1924, pp. 1–156.

[2] A detailed exposition of the theory at connectivity mod 2 is given in Veblen's *Cambridge Colloquium Lectures*. See also Veblen, *Decomposition of N-Space by a Polyhedron, Trans. Amer. Math. Soc.*, **14**, 65–72 (1913).

ASSOCIATED TYPES OF LINEAR CONNECTION

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1. *Introduction.*—Let us consider the components of affine connection which are a set of quantities $\Gamma^i_{r\lambda}$, functions of the coördinates $u^1, \ldots, u^n$, of a manifold, symmetric in r and λ. When these functions are given a geometry is determined.[1] A new geometry is determined if the quantities $\Gamma^i_{r\lambda}$ are replaced by another set $\bar{\Gamma}^i_{r\lambda}$.

If we transform from the quantities Γ to the quantities $\bar{\Gamma}$ according to some definite law it may be expected that the corresponding geometries also will be related in a definite way. In this paper it is proposed to study the geometries associated with one another by the transformations

$$\bar{\Gamma}^s_{i\lambda} = A^s_\alpha \left(\frac{\partial a^\alpha_i}{\partial u^\lambda} + a^r_i \, \Gamma^\alpha_{r\lambda} \right) \tag{1}$$

where $\| a^i_j \|$ is a nonsingular matrix whose elements are functions of the coördinates and A^s_α is the co-factor of the element a^s_α divided by the determinant $| a^i_j | = a$.

2. *Resultant of Two Transformations.*—If the transformation associated with the matrix a^i_j is followed by the transformation associated with the matrix b^i_j we have for the resultant transformation on the functions Γ,

$$\begin{aligned} \bar{\bar{\Gamma}}^s_{i\lambda} &= B^s_\alpha \left(\frac{\partial b^\alpha_i}{\partial u^\lambda} + b^r_i \, \bar{\Gamma}^\alpha_{r\lambda} \right) \\ &= B^s_\alpha \left(\frac{\partial b^\alpha_i}{\partial u^\lambda} + b^r_i \, A^\alpha_\beta \left[\frac{\partial a^\beta_r}{\partial u^\lambda} + a^\rho_r \, \Gamma^\beta_{\rho\lambda} \right] \right) \\ &= B^s_\alpha \frac{\partial b^\alpha_i}{\partial u^\lambda} + B^s_\alpha \, b^r_i \, A^\alpha_\beta \frac{\partial a^\rho_r}{\partial u^\lambda} + B^s_\alpha \, b^r_i \, A^\alpha_\beta \, a^\rho_r \, \Gamma^\beta_{\rho\lambda} \end{aligned} \tag{2}$$

The matrices determined by a^i_j, b^i_j, and $c^i_j = b^r_j \, a^i_r$ are easily seen to satisfy the following relations:

$$B^k_l \, c^j_k = a^j_l, \; A^r_j \, c^j_k = b^r_k, \; c^r_j \, a^j_l = B^r_l, \; B^r_l \, A^l_j \, c^j_k = \delta^r_k, \; B^r_l \, A^l_j = C^r_j.$$

From the equation $c^j_k = b^r_k \, a^j_r$ we obtain by differentiation,

$$\frac{\partial c^j_k}{\partial u^\lambda} = b^r_k \frac{\partial a^j_r}{\partial u^\lambda} + a^j_r \frac{\partial b^r_k}{\partial u^\lambda}$$

and therefore

$$\begin{aligned} C^s_j \frac{\partial c^j_k}{\partial u^\lambda} &= C^s_j \, b^r_k \frac{\partial a^j_r}{\partial u^\lambda} + C^s_j \, a^j_r \frac{\partial b^r_k}{\partial u^\lambda} \\ &= C^s_j \, b^r_k \frac{\partial a^j_r}{\partial u^\lambda} + B^s_r \frac{\partial b^r_k}{\partial u^\lambda} \end{aligned} \tag{3}$$

hence

$$B^s_\alpha \frac{\partial b^\alpha_i}{\partial u^\lambda} = C^s_\alpha \frac{\partial c^\alpha_i}{\partial u^\lambda} - C^s_\alpha b^r_i \frac{\partial a^\alpha_r}{\partial u^\lambda}.$$

If this is substituted in the last member of equation (2) the expression for $\bar{\bar{\Gamma}}^s_{i\lambda}$ reduces to[2]

$$\bar{\bar{\Gamma}}^s_{i\lambda} = C^s_\alpha \left(\frac{\partial c^\alpha_i}{\partial u^\lambda} + c^r_i \Gamma^\alpha_{r\lambda} \right). \tag{4}$$

This shows that the resultant of two transformations on the functions Γ is the transformation associated with the resultant of the two corresponding matrices.

3. *The Inverse Transformation.*—If in the previous § the transformation b^i_j is the inverse of A^i_j the transformation c^i_j reduces to the identity, consequently

$$c^r_i = \delta^r_i, \qquad C^s_\alpha = \delta^s_\alpha, \qquad \frac{\partial c^\alpha_i}{\partial u^\lambda} = 0$$

and the transformation on the functions Γ also reduces to the identity. We give, however, a separate treatment of the relation between an associated transformation and its inverse.

The transformation on the functions Γ associated with the matrix A^j_i is

$$\Gamma^s_{m\lambda} = \left(\frac{\partial A^\alpha_m}{\partial u^\lambda} + A^r_m \bar{\Gamma}^\alpha_{r\lambda} \right) a^s_\alpha. \tag{5}$$

This relation between Γ and $\bar{\Gamma}$ should be the same as that given by equations (2).

We solve equations (2) for $\Gamma^t_{r\lambda}$ by multiplying through by A^m_t and summing on the index m.

$$\Gamma^t_{l\lambda} = a^t_s A^m_l \bar{\Gamma}^s_{m\lambda} - A^m_l \frac{\partial a^t_m}{\partial u^\lambda}.$$

From the relation $A^m_l a^t_m = \delta^t_l$ we obtain

$$a^t_m \frac{\partial A^m_l}{\partial u^\lambda} = - A^m_l \frac{\partial a^t_m}{\partial u^\lambda}.$$

The equations for $\Gamma^t_{l\lambda}$ then become

$$\Gamma^t_{l\lambda} = a^t_s A^m_l \bar{\Gamma}^s_{m\lambda} + a^t_m \frac{\partial A^m_l}{\partial u^\lambda}$$

and these are the same as equations (5).

It follows that the transformation (5) may be taken as the definition of the inverse of the transformation (2).

It is, of course, possible that there should be more than one transformation taking a given set of functions Γ into themselves. The totality of

such transformations clearly form a group. Only the one associated with the identical matrix $\| \delta_j^i \|$ will be regarded as the identity.

It may be noticed, in passing, that if G is the group of transformations leaving Γ_{ij}^k invariant and if $S\Gamma_{ij}^k = \bar{\Gamma}_{ij}^k$ then SGS^{-1} is the group of transformations leaving $\bar{\Gamma}_{ij}^k$ invariant.

4. *Invariant Relations.*—We next consider the relations between the curvature tensor formed from the original functions Γ and the corresponding tensor formed from the new Γ's. We regard equations (2) as differential equations in the a's and write them in the form

$$\frac{\partial a_m^t}{\partial u^\lambda} = a_s^t \bar{\Gamma}_{m\lambda}^s - a_m^r \Gamma_{r\lambda}^t \tag{6}$$

The integrability conditions for these equations are

$$\begin{aligned} a_s^t \left[\frac{\partial \bar{\Gamma}_{m\lambda}^s}{\partial u^\mu} - \frac{\partial \bar{\Gamma}_{m\mu}^s}{\partial u^\lambda} + \bar{\Gamma}_{m\lambda}^r \bar{\Gamma}_{r\mu}^s - \bar{\Gamma}_{m\mu}^r \bar{\Gamma}_{r\lambda}^s \right] \\ = a_m^r \left[\frac{\partial \Gamma_{r\lambda}^t}{\partial u^\mu} - \frac{\partial \Gamma_{r\mu}^t}{\partial u^\lambda} + \Gamma_{r\lambda}^s \Gamma_{s\mu}^t - \Gamma_{r\mu}^s \Gamma_{s\lambda}^t \right] \end{aligned} \tag{7}$$

which may also be written

$$a_s^t \bar{R}_{m\lambda\mu}^s = a_m^r R_{r\lambda\mu}^t \tag{8}$$

The quantities $R_{m\lambda\mu}^s$ are the components of the curvature tensor, and equations (8) enable us to compute the components of the curvature tensor in the associated geometry from the coefficients of the transformation and the components of the original tensor.

For fixed values of μ and λ equations (8) form a system of n^2 homogeneous linear equations in the n^2 quantities a_j^i. It follows that for values of a_j^i which are not all zero the determinant of the coefficients must vanish. The equations thus obtained connecting the $R_{m\lambda\mu}^s$ with the $\bar{R}_{m\lambda\mu}^s$ will have the same form for all associated geometries. There are $n\frac{(n-1)}{2}$ of these invariant relations.

The system of equations (8) may be solved for the $\bar{R}_{m\lambda\mu}^s$. The resulting formulas are $\bar{R}_{m\lambda\mu}^\rho = A_t^\rho a_m^r R_{r\lambda\mu}^t$. By contraction on ρ and m we obtain $\bar{R}_{m\lambda\mu}^m = R_{m\lambda\mu}^m$ and hence the contracted tensor $R_{m\lambda\mu}^m$ is an invariant under the transformations (1). It follows that equi-affine geometries are transformed into equi-affine geometries.[3]

5. *Special Geometries.*—We consider in this § some special cases in which the quantities $R_{m\lambda\mu}^s$ and the Γ's satisfy certain relations. The simplest case of this kind is the case in which all of the components $R_{m\lambda\mu}^s$ vanish. Equations (8) in this case reduce to $a_s^t \bar{R}_{m\lambda\mu}^s = 0$; it follows that $A_t^\rho a_s^t \bar{R}_{m\lambda\mu}^s = \bar{R}_{m\lambda\mu}^\rho = 0$ and we have the result

If all the components of the curvature tensor $R^s_{m\lambda\mu}$ vanish in any given geometry the corresponding components vanish in all associated geometries.

This result can be extended. By taking derivatives of both sides of (8) with respect to u^j we obtain

$$a^t_s \frac{\partial \bar{R}^s_{m\lambda\mu}}{\partial u^j} + \frac{\partial a^t_s}{\partial u^j} \bar{R}^s_{m\lambda\mu} = a^r_m \frac{\partial R^t_{r\lambda\mu}}{\partial u^j} + \frac{\partial a^r_m}{\partial u^j} R^t_{r\lambda\mu} \tag{9}$$

By substituting for $\frac{\partial a^i_k}{\partial u^j}$ and making use of (8) this reduces to

$$\begin{aligned} & a^t_s \left[\frac{\partial \bar{R}^s_{m\lambda\mu}}{\partial u^j} + \bar{R}^\sigma_{m\lambda\mu} \bar{\Gamma}^s_{\sigma j} - \bar{R}^s_{\sigma\lambda\mu} \bar{\Gamma}^\sigma_{mj} \right] \\ & = a^r_m \left[\frac{\partial R^t_{r\lambda\mu}}{\partial u^j} + R^\rho_{r\lambda\mu} \Gamma^t_{\rho j} - R^t_{\rho\lambda\mu} \Gamma^\rho_{rj} \right] \end{aligned} \tag{10}$$

or

$$a^t_s \bar{T}^s_{m\lambda\mu j} = a^r_m T^t_{r\lambda\mu j} \tag{11}$$

By the same argument as that used above it is found that

If the quantities T vanish in any given geometry the corresponding quantities vanish in all associated geometries.

The process may obviously be repeated.

It is clear that there also exist invariant relations connecting the quantities T and $\bar{T}$ of the same type as those connecting R and $\bar{R}$ and that $T^m_{m\lambda\mu j}$ obtained by contraction is invariant.

6. *Change of Coördinates.*—If the coördinates are changed by substituting for the u^i functions of $u'^1, \ldots, u'^n$ the functions Γ^i_{ij} are transformed into $'\Gamma^i_{ij}$ according to the law

$$'\Gamma^\lambda_{\mu\nu} = B^\lambda_i b^j_\nu \left(\frac{\partial b^i_\mu}{\partial x^j} + b^i_\mu \Gamma^i_{ij} \right) \tag{12}$$

where $b^i_j = \frac{\partial u^i}{\partial u'^j}$ and $B^i_j = \frac{\partial u'^i}{\partial u^j}$.

The functions $\bar{\Gamma}^k_{ij}$ undergo the same transformation into the functions $'\bar{\Gamma}^k_{ij}$.

We now inquire whether there exists, in the coördinate system u'^i a transformation of the form

$$'\bar{\Gamma}^s_{i\lambda} = 'A^s_\alpha \left(\frac{\partial' a^\alpha_i}{\partial u'^\lambda} + 'a^r_i \,'\Gamma^\alpha_{r\lambda} \right). \tag{13}$$

It is not to be expected that the transformation, if it exists, should in all cases be unique.

From (12) applied to Γ we have

$$'\bar{\Gamma}^s_{i\lambda} = B^s_\alpha b^j_\lambda \left(\frac{\partial b^\alpha_i}{\partial u^j} + b^\sigma_i \bar{\Gamma}^\alpha_{\sigma j} \right) = B^s_\alpha b^j_\lambda \left[\frac{\partial b^\alpha_i}{\partial u^j} + b^\sigma_i A^\alpha_\beta \left(\frac{\partial a^\beta_\sigma}{\partial u^j} + a^\rho_\sigma \Gamma^\beta_{\rho j} \right) \right]. \tag{14}$$

Also if there is to be a transformation (13)

$$
\begin{aligned}
{}'\bar{\Gamma}^{s}_{i\lambda} &= {}'A^{s}_{\alpha}\left(\frac{\partial' a^{\alpha}_{i}}{\partial u'^{\lambda}} + {}'a^{\sigma}_{i}\,{}'\Gamma^{\alpha}_{\sigma\lambda}\right) \\
&= {}'A^{s}_{\alpha}\left[\frac{\partial' a^{\alpha}_{i}}{\partial u'^{\lambda}} + {}'a^{\sigma}_{i}\,B^{\alpha}_{\beta}\,b^{j}_{\lambda}\left(\frac{\partial b^{\beta}_{\sigma}}{\partial u^{j}} + b^{\rho}_{\sigma}\,\Gamma^{\beta}_{\rho j}\right)\right].
\end{aligned}
\tag{15}
$$

In order that these two formulas for the ${}'\bar{\Gamma}^{s}_{i\lambda}$ should agree it is sufficient that the two sets of terms containing $\Gamma^{\beta}_{\rho j}$ on the right should agree and that the two sets free from the Γ's should agree. The first of these conditions is

$$
{}'A^{s}_{\alpha}\,{}'a^{\sigma}_{i}\,B^{\alpha}_{\beta}\,b^{j}_{\lambda}\,b^{\rho}_{\sigma}\,\Gamma^{\beta}_{\rho j} = B^{s}_{\alpha}\,b^{j}_{\lambda}\,b^{\sigma}_{i}\,A^{\alpha}_{\beta}\,a^{\rho}_{\sigma}\,\Gamma^{\beta}_{\rho j} \tag{16}
$$

and this is clearly satisfied if

$$
{}'a^{\mu}_{i} = b^{\gamma}_{i}\,B^{\mu}_{\rho}\,a^{\rho}_{\sigma}. \qquad {}'A^{s}_{\lambda} = b^{\beta}_{\lambda}\,B^{s}_{\alpha}\,A^{\alpha}_{\beta}. \tag{17}
$$

If we make use of these formulas and at the same time replace $\dfrac{\partial' \alpha^{\alpha}_{i}}{\partial u'^{\lambda}}$ by $\dfrac{\partial' a^{\alpha}_{i}}{\partial u^{j}}\,b^{j}_{\lambda}$ the second condition takes the form

$$
\begin{aligned}
B^{s}_{\alpha}\,b^{j}_{\lambda}\frac{\partial b^{\alpha}_{i}}{\partial u^{j}} + B^{s}_{\alpha}\,b^{j}_{\lambda}\,b^{\sigma}_{i}\,A^{\alpha}_{\beta}\frac{\partial a^{\beta}_{\sigma}}{\partial u^{j}} &= b^{\beta}_{\alpha}\,B^{s}_{\gamma}\,A^{\gamma}_{\beta}\,b^{j}_{\lambda}\frac{\partial' a^{\alpha}_{i}}{\partial u^{j}} \\
&\quad + b^{l}_{\alpha}\,B^{s}_{\gamma}\,A^{\gamma}_{l}\,b^{\sigma}_{i}\,B^{\rho}_{\mu}\,A^{\mu}_{\sigma}\,B^{\alpha}_{\beta}\,b^{j}_{\lambda}\frac{\partial b^{\beta}_{\rho}}{\partial u^{j}}.
\end{aligned}
\tag{18}
$$

When the value of $\dfrac{\partial' a^{\alpha}_{i}}{\partial u^{j}}$ determined from ${}'a^{\alpha}_{i} = b^{\sigma}_{i}\,B^{\alpha}_{\rho}\,a^{\rho}_{\sigma}$ is substituted in the first term on the right this condition is found to be identically satisfied. Hence the transformation associated with the matrix ${}'a^{i}_{j}$ defined by (17) transforms the functions ${}'\Gamma^{\mu}_{ij}$ into ${}'\bar{\Gamma}^{\mu}_{ij}$.

[1] For the manner in which the quantities $\Gamma^{i}_{r\lambda}$ determine a geometry see Eisenhart and Veblen, "The Riemann Geometery and Its Generalizations," *Proc. Nat. Acad. Sci.*, **8** (1922), p. 19.

[2] In case the functions Γ^{k}_{ij} are not symmetric in i and j a second group of transformations may be defined by

$$
\bar{\Gamma}^{s}_{\mu\lambda} = A^{s}_{\alpha}\left(\frac{\partial a^{\alpha}_{\lambda}}{\partial u^{\mu}} + a^{\gamma}_{\lambda}\,\Gamma^{\alpha}_{\mu\gamma}\right).
$$

[3] Cf. Veblen, "Equiaffine Geometry of Paths," *Proc. Nat. Acad. Sci.*, **9** (1923), p. 3.

CONFORMAL CORRESPONDENCE OF RIEMANN SPACES

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1. In this paper is pointed out the existence of a set of quantities which have the same values at corresponding points of two Riemann spaces mapped conformally on each other. They are analogous to the components of the projective connection[1] for the geometry of paths, and are of importance in the determination of conformal invariants.[2] In particular, the conformal curvature tensor[3] is here obtained by expressing integrability conditions of their law of transformation.[4] Finally a geometrical interpretation is given to them by means of the idea of parallel displacement of a vector.

2. A Riemann space with fundamental tensor g'_{ij} is conformally represented on another such space with fundamental tensor g_{ij} when

$$g'_{ij} = \sigma g_{ij},$$

σ being a point function, and corresponding points having the same coordinates. Under these conditions we have also that

$$g'^{ij} = \frac{1}{\sigma} g^{ij}.$$

The quantities $g^{ij}\, g_{kl}$ are therefore the components of a conformal tensor.

It is well known[5] that the Christoffel symbols of the second kind for the two spaces are connected by the relations

$$\left\{ {i \atop jk} \right\}' = \left\{ {i \atop jk} \right\} + \frac{1}{2} (\delta^i_j\, s_k + \delta^i_k\, s_j - g^{i\alpha} g_{jk} s_\alpha), \qquad (2.1)$$

where

$$s_k = \frac{\partial \log \sigma}{\partial x^k}.$$

Contraction for i and j in (2.1) gives

$$s_k = \frac{2}{n} \left(\left\{ {\alpha \atop \alpha k} \right\}' - \left\{ {\alpha \atop \alpha k} \right\} \right). \qquad (2.2)$$

Substitution from (2.2) in (2.1) and transposition show that the quantities

$$K^i_{jk} = \left\{ {i \atop jk} \right\} - \frac{\delta^i_j}{n} \left\{ {\alpha \atop \alpha k} \right\} - \frac{\delta^i_k}{n} \left\{ {\alpha \atop \alpha j} \right\} + \frac{1}{n} g^{i\beta} g_{jk} \left\{ {\alpha \atop \alpha \beta} \right\} \qquad (2.3)$$

have the same values at corresponding points in both spaces. They will be called the *components of the conformal connection*. From their definition follows

$$K^{\alpha}_{\alpha j} = K^{\alpha}_{j\alpha} = 0.$$

Under an analytic transformation of coördinates the K's have the following law of transformation

$$\frac{\partial \bar{x}^i}{\partial x^\alpha} K^{\alpha}_{jk} = \bar{K}^{i}_{\alpha\beta} \frac{\partial \bar{x}^\alpha}{\partial x^j} \frac{\partial \bar{x}^\beta}{\partial x^k} + \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} + \frac{\partial \bar{x}^i}{\partial x^j} \psi_k + \frac{\partial \bar{x}^i}{\partial x^k} \psi_j - g^{\alpha\beta} g_{jk} \frac{\partial \bar{x}^i}{\partial x^\alpha} \psi_\beta, \quad (2.4)$$

where

$$\psi_j = \frac{1}{n} \frac{\partial \log \Delta (x, \bar{x})}{\partial x^j}, \qquad \Delta(x, \bar{x}) = \left| \frac{\partial x^i}{\partial \bar{x}^j} \right|.$$

From the vanishing of the covariant derivatives of $g^{\alpha\beta}$ and g_{jk} we find by use of (2.3) that

$$\frac{\partial}{\partial x^l} (g^{\alpha\beta} g_{jk}) = g^{\alpha\beta} g_{\gamma k} K^{\gamma}_{jl} + g^{\alpha\beta} g_{j\gamma} K^{\gamma}_{kl} - g^{\gamma\beta} g_{jk} K^{\alpha}_{\gamma l} - g^{\alpha\gamma} g_{jk} K^{\beta}_{\gamma l}. \quad (2.5)$$

Differentiation of (2.4) with respect to x^l, substitution from (2.4) and (2.5), interchange of k and l, and subtraction give

$$F^{i}_{jkl} = \bar{F}^{\alpha}_{\beta\gamma\delta} \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\beta}{\partial x^j} \frac{\partial \bar{x}^\gamma}{\partial x^k} \frac{\partial \bar{x}^\delta}{\partial x^l} + \delta^i_l c_{jk} - \delta^i_k c_{jl} + g_{jk} c^i_l - g_{jl} c^i_k + (\delta^i_l g_{jk} - \delta^i_k g_{jl}) \psi_\alpha \psi^\alpha, \quad (2.6)$$

where F^{i}_{jkl} is formed from the K's in the same way that the Riemann curvature tensor is formed from the Christoffel symbols of the second kind, and

$$c_{jk} = K^{\alpha}_{jk} \psi_\alpha - \psi_j \psi_k - \frac{\partial \psi_j}{\partial x^k}, \qquad c^i_l = g^{i\alpha} c_{\alpha l}.$$

Contraction for i and l in (2.6) gives

$$F_{jk} = \bar{F}_{\alpha\beta} \frac{\partial \bar{x}^\alpha}{\partial x^j} \frac{\partial \bar{x}^\beta}{\partial x^k} + (n-2) c_{jk} + g_{jk} c + (n-1) g_{jk} \psi_\alpha \psi^\alpha, \quad (2.7)$$

where $c^{\alpha}_{\alpha} = c$. Multiplying by g^{jk} and summing, we get

$$F = \bar{F} + 2(n-1)c + n(n-1) \psi_\alpha \psi^\alpha. \quad (2.8)$$

When c_{jk} is eliminated from (2.6) by means of (2.7) and (2.8), it is found that $\psi_\alpha \psi^\alpha$ cancels out and that the quantities

$$(n-2) F^{i}_{jkl} + \delta^i_k F_{jl} - \delta^i_l F_{jk} + g_{jl} F^{i}_{k} - g_{jk} F^{i}_{l} + \frac{F}{n-1} (\delta^i_l g_{jk} - \delta^i_k g_{jl})$$

are components of a tensor, which is conformal in character because it is expressed in terms of the K's and the tensor $g^{ij}\,g_{kl}$. It is the conformal tensor discovered by Weyl.[6] The above expression is, moreover, identical with that obtained from the ordinary expression for this tensor by replacing each Christoffel symbol by the corresponding component of the conformal connection.

3. The equations

$$x^i = x_o^i + \bar{x}^i - \tfrac{1}{2}\,(K^i_{\alpha\beta})_0\,\bar{x}^\alpha\,\bar{x}^\beta \qquad (3.1)$$

define a transformation of coördinates such that at the point $x^i = x_o^i$

$$\frac{\partial x^i}{\partial \bar{x}^j} = \frac{\partial \bar{x}^i}{\partial x^j} = \delta^i_j, \qquad \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} = K^i_{jk}, \qquad \psi_j = 0.$$

Hence from (2.4) it is seen that the components of the conformal connection vanish in the $\bar{x}$ coördinate system at the origin. Consider now a vector $d\bar{x}^i$ at the origin. By parallel displacement to the point whose coordinates are $d\bar{x}^i$ its components become[7]

$$d\bar{x}^i - \overline{\{^{\,i}_{jk}\}}_0\,d\bar{x}^j\,d\bar{x}^k,$$

or, by use of (2.3)

$$d\bar{x}^i - 2d\bar{x}^i\,a_j\,d\bar{x}^j + a^i\,\bar{g}_{jk}\,d\bar{x}^j\,d\bar{x}^k,$$

where $na_i = \overline{\{^{\,\alpha}_{\alpha i}\}}_0$ and $a^i = \bar{g}^{i\alpha}\,a_\alpha$. If the vector is taken tangent to a minimal curve through the origin, then

$$\bar{g}_{jk}\,d\bar{x}^j\,d\bar{x}^k = 0,$$

and in the $\bar{x}$ coördinate system the original and displaced vectors have proportional components.

4. It is interesting to note that if all the terms are transposed to the left in (2.5), the left-hand member is then formed from the conformal tensor $g^{\alpha\beta}\,g_{jk}$ in the same way as its covariant derivative with the components of the conformal connection replacing the corresponding Christoffel symbols. Also from (2.5) it follows that in the special coördinate system defined by (3.1) the components of the tensor $g^{\alpha\beta}\,g_{jk}$ are stationary at the origin.

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[1] Cf. T. Y. Thomas, these PROCEEDINGS, April, 1925.

[2] Cf. Veblen and J. M. Thomas, these PROCEEDINGS, April, 1925.

[3] Weyl, *Math. Zeit.*, 1918, 2, p. 404.

[4] Cf. J. M. Thomas, these PROCEEDINGS, April, 1925, for a similar discussion of the projective tensor.

[5] Cf. Schouten, *der Ricci-Kalkül*, p. 168.

[6] *Loc. cit.*, Actually, Weyl's tensor is minus the above.

[7] Cf. Weyl, *Raum, Zeit, Materie*, 4th ed., p. 101.

PINHOLE PROBE MEASUREMENTS OF THE PHASE CHANGE OF THE TELEPHONIC END PLATES, ACTING ON A CLOSED CYLINDRICAL AIR COLUMN IN LONGITUDINAL ACOUSTIC VIBRATION

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1. *Circuits.*—Measurements of nodal pressures indicated by the pinhole probe are made with the U-tube interferometer. Antinodes do not respond. The graphs show the fringe displacements, s under varying conditions.

The trend of the work may most easily be shown by describing the successive circuits used and the acoustic effects observed with each. In figure 1*a*, E is the primary electromotive force (one or two storage cells) sending a current in series through the two telephones T and T' and the resistance R. This current is periodically broken at the brushes b, by the plate commutator B, rotated at controllable speed by a small motor (not shown). Thus, both telephones give out the same note, the pitch being determined by the angular velocity of B. One of the telephones, T', is provided with a switch S, so that its current may be reversed.

The two telephones are connected at the mouth-piece, by the tube tt, originally 15 in. long (effectively), and 1.5 cm. in diam. At right angles to it and at its center, the two pinhole probes (salient s, and reëntrant r,) are inserted and s is joined to the one shank of the U-gauge by a short end of rubber tubing. Joints are sealed with wax. This circuit (1*a*), in series admits of the more useful modification with the telephone circuits in parallel. Additional resistances, inductances, capacities, may then be inserted in branch T or T'.

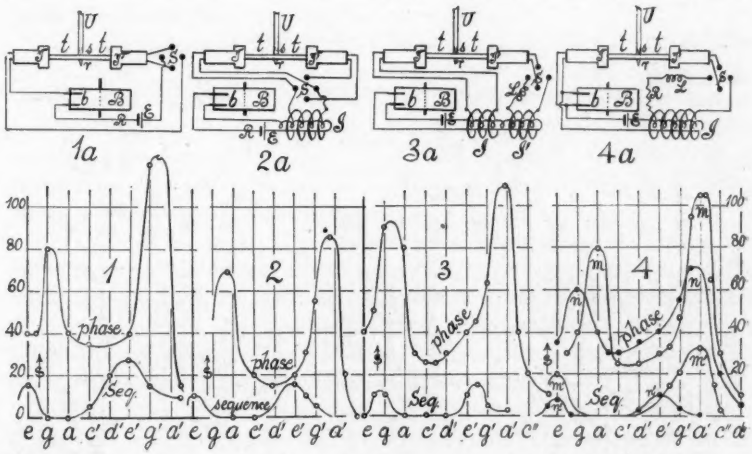
In figure 2*a*, a small inductor I has been introduced, cell E feeding the primary. The telephone T and T' are now operated by the induced current with a switch at S for one of them. A resistance at R is convenient. In the arrangement, figure 3*a*, two secondaries, I and I', are wound on the same primary, the other adjustments remaining as before.

In a final arrangement the primary circuit from E actuates the telephone T directly, whereas the secondary circuit from I, with a switch S, passes through the telephone T'. This has been improved in figure 4*a* where a branch circuit from the primary passes through the telephone T. The advantage is the easy equalization of telephone currents by a resistance R placed in either circuit, additional inductance being added at L.

2. *Survey in Pitch.*—The graphs obtained with each of the circuit designs, showing the fringe displacement s (nodal intensity) as the pitch

gradually rises from c (four foot octave) to a′ (two foot octave) are given below the corresponding diagrams of circuits. With the telephone plates in phase (i.e., the plates moving outward or inward together) there is marked response (node) near a′ and again near a; above a′ the tube is silent indefinitely. Between a′ and a there is here apt to be multiresonance, so that the air column vibrates at all pitch intervals. The saliency at a′, although usually very pronounced (figs. 1, 2, 3) is sometimes reduced as in figure 4, n, n' possibly owing to inherent differences of phase in the two circuits; but probably incidental, as the curves m, m', obtained with improved telephones, imply.

When the current in the telephone T' is reversed by the switch S, the plates vibrate inward and outward, respectively, i.e., in sequence. The

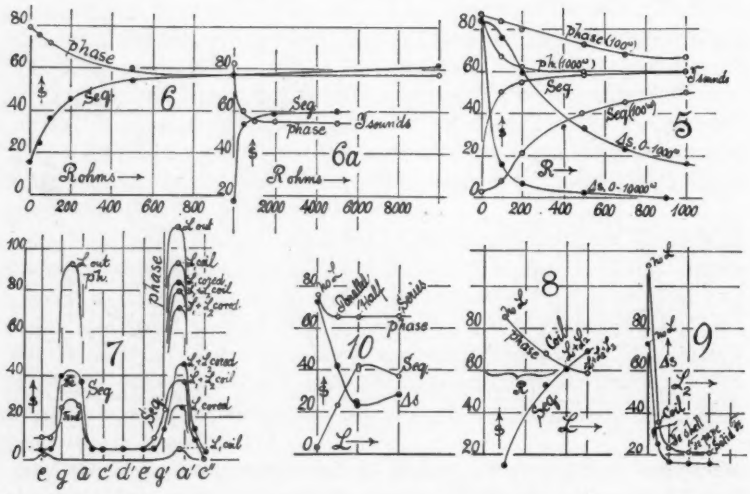


nodes thus vanish from the middle of the tube and appear at the ends. There is therefore no middle pinhole pressure (s) at a, a′. Curiously enough there usually was response (middle node) in this case at e, e′, sometimes quite marked, as in figure 1. It was subsequently traced to an inequality in the mouth-piece of the telephones and removed as in figure 4, m'.

3. *Continuous Change of Phase Difference.*—This may be brought about by inserting increasing resistance R, in one of the duplicate telephone circuits, leaving the other unchanged. The first test was made with the design of figure 3 and the results are shown in figure 5. Between 0 and 1000 ohms, the fringe displacement, s, are given in steps of 100 ohms; between 1000 and 10,000 ohms, in steps of 1000 ohms. The phase displacement graph falls while the sequence graph rises at a rapid rate, continually. From the initial difference of $\Delta s = 87 - 3 = 84$ at $R = 0$ to the final

difference of $\Delta s = 60 - 60 = 0$ at $R = 10{,}000$ ohms, the curves have become a symptotically coincident. The fall of (middle) nodal intensity from $s = 87$ to $s = 60$ is due to the fact that in the latter case but a single telephone plate, T, vibrates, whereas the other reflects like a rigid wall. In the former case both plates vibrate, reinforcing each other. As a whole we have in the sequence graph an interesting exhibit of a *continuous change of the type of reflection;* initially without change of phase, without middle node ($s = 3$) it passes eventually into reflection with change of phase and a strong middle node ($s = 60$). The different graphs offer a means of converting any Δs observed, into the corresponding excess resistance R.

4. *Phase Difference Passing through Zero.*—In figure 5 one telephone is silenced before appreciable phase reversal occurs, possibly because the ini-



tial resistances, etc., in the T and T' circuits (fig. 3) are large. The trial was therefore repeated with the adjustment like figure 1 but with T, T' circuits in parallel, T' containing R and S. An example of the results is given in figures 6 and 6*a*. The curves now intersect at 1000 ohms when one of the telephone plates is *effectively* at rest. Thereafter this telephone (T') tends to vibrate more and more in phase again; but the relatively high resistances soon cut the modified circuit out. One notes that the sequence curve still rises with increasing R after the phase graph has ceased to fall. T' vibrating in sequence is probably more sensitive to current than when vibrating in phase. At all events the plate of T' seems to reflect like a wall, while still vibrating in sequence and thus passes into the phase condition unsymmetrically.

5. *Phase Reversal Owing to Inductance.*—After this experiments were made with capacities (C) and inductances (L) with the hope of observing more striking phase reversals. I give a few examples in figure 7, the empty coils being first inserted into one telephone circuit (T' fig. 3) and the iron core thereafter, the steps of inductance L being estimated at 5 Henrys, though they were far smaller at the frequency a′ (440), as the cores of iron were not laminated. The survey in pitch carried out in the graph at each L is in conformity with the unloaded graph.

Figure 8 is a summarized example of the fringe displacements s found (adjustment fig. 1 in parallel) after the insertion of the coils (resistance R) and after the subsequent insertion of the iron cores (L_1, L_2, L_3). In these cases also the phase curve generally tends to become horizontal while (after the intersection) the sequence curve still rises.

An incidental result was obtained in comparing hollow and solid iron cores. The L-effect in the two cases was about the same. Thus in inserting an inch solid rod into the coil L_2 the Δs observed with and without iron was not appreciably larger than when the same length of inch gas pipe or even a tin-plate pipe (diam. 1 inch) was inserted. Thus the iron at frequencies above 400 per sec. here exhibits a magnetic skin or shielding effect. Examples are given in figure 9.

6. *Data.*—If we regard the fringe displacements Δs as indicating the effective or virtual current in the telephone, a few tentative tests may be begun using a chart like figure 5 for converting Δs into the excess resistance R in circuit, E, R, L, C, ω having their usual meaning. Hence if we temporarily disregard as figure 5 seems to permit, the inherent phase differences, the telephone amplitude may be written $E/\sqrt{R^2 + (1/C\omega - L\omega)^2}$.

If the same reduction of Δs is produced in one case by additional resistance (ΔR, coils only) and in the other by additional inductance (ΔL iron core in) from the same initial Δs we may deduce $\Delta R^2/\Delta L\omega = \omega(L' + L) - 2/C\omega$. For another coil L_1, R_1, of the same C and ω one would get by comparison

$$\omega(L + L' - L_1 - L_1') = \Delta R_1^2/\omega\Delta L_1 - \Delta R^2/\omega\Delta L$$

If, therefore, it were permissible to neglect the initial inductance ($L = L_1 = 0$) of the coils, without iron,

$$\pm \omega L_1' + \Delta R_1^2/\omega L_1' = \pm \omega L' + \Delta R^2/\omega L' = \text{const.}$$

so that L_1' is given in terms of a standard L'. The equation does not work out well, implying that some of the postulates are prohibited. If capacity C is excluded, the case ($\Delta L^2 = \Delta R^2/w^2 = L_1'^2$ nearly) is simpler and the results more trustworthy; but if is probably better to make the complete computation. Allowing for the phases this is $\Delta L^2 = (R^2/L^2\omega^2)\Delta R^2/\omega^2$.

7. *Initial Phase Reversals.*—The data of figure 10 were obtained by the method of figure 1 (in parallel) the full fringe displacement in the absence of coils being $\Delta s = 65$.

The coil examined was a little induction coil, consisting of two identical half secondaries with fixed iron. So far as the phase graph is concerned it makes little difference how the layers are taken. The sequence graph rises naturally when the two layers are combined in parallel and then inserted separately; but curiously enough each single layer shows greater impedence alone than when they are taken together in series. Since the impedence of the half coil (*caet par.*) cannot be greater than the whole, the half coils were considered as associated with an initial phase increment which later changes sign. In other adjustments initial phase differences may often be passed through zero. The subject will be treated specifically in a subsequent note.

ON THE BOLTZMANN EQUATION $\rho = \rho_0 \exp(-w/kt)$

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Read before the Academy on April 28, 1925

In a canonical ensemble the densities D of the systems in their phase-space satisfy the condition $D = D_0 \exp(-\Delta\epsilon/kt)$, where $\Delta\epsilon = \epsilon - \epsilon_0$ is the change in the dynamical energy (kinetic and potential) from the assumed original to any arbitrary phase. In case the system is made up of point-masses with no mutual potential energy, integration leads to the equation in densities of matter as $\rho = \rho_0 \exp(-w/kt)$ where w is the change in potential energy or work done against the external field (there being no internal field and the kinetic energy being constant, see, e.g., Cl. Schaefer, "Einführing in die theoretische Physik," Bd. 2. T. 1, pp. 464–8). This latter convenient form of the equation is itself sometimes regarded as of rather wide applicability, as a sort of thermodynamic theorem, and is applied, for example, to the "evaporation" of electrons from metals in thermionic emission to estimate the relative spacial densities of electrons within and without the metal, w being the "work done in taking the electron through the surface of the metal." Inasmuch as kt has the dimensions of work it is of course possible to define by the equation $w = kt \log(\rho/\rho_0)$ the "work done" and thus have an identity by definition; but then it is necessary to find a useful interpretation of w in terms of ordinary conceptions of work (mechanical, thermodynamic or electrical) if the identity is to possess physical significance. In the case of electron evaporation

neither w nor ρ_0 is readily accessible to direct measurement and hence direct comparison between the formula the physical fact is doubly undetermined.

For a perfect gas subject to an external field of force the equation may be proved true by hydromechanics, with w possessed of its ordinary significance, as is fore-shadowed by the statistical demonstration. For this case, however, one may equally well write $w = -kt \log (p/p_0)$ or $w = kt \log (v/v_0)$, though the transformations are not universally valid except on redefinition of w. So, too, in place of w one may write $\Delta\epsilon$ (estimated per molecule) or $\Delta\chi$ (estimated per molecule with $\chi = \epsilon + pv$, the total heat, since pv is constant), or again one may write the equations as $\Delta\zeta = 0$ with $\zeta = \epsilon + pv - t\eta$, or $\Delta\psi = 0$ with $\psi = \epsilon - t\eta$, since $\Delta\eta = k \log (v/v_0)$ per molecule; but if the attempt be made to generalize from the perfect gas, the various equations (or definitions of w) became diverse. For example, in the gravity field if we take $w = mgz$, we have

$$mgz = -kt \log (\rho/\rho_0), \qquad mgdz = -ktd\rho/\rho \qquad dp = -\rho g dz$$
$$mdp = ktd\rho \qquad p = kt/m.\ (\rho + C).$$

With $C = 0$ this is the equation of state of an ideal gas, with $C \neq 0$ it is not a commonly found equation of state. The equation $mgz = -kt \log (p/p_0)$ leads to $mp = kt\rho$ without the constant C. Thus for a liquid or a real gas (van der Waals' or Keyes' equation) the interpretation of w cannot be mgz or else the function cannot be log (ρ/ρ_0) or log (p/p_0). If we turn to the evaporation of a liquid into a vapor with equilibrium between the liquid and vapor phases, we see that $w = kt \log (p/p_0)$ would give $w = 0$ whereas $w = -kt \log (\rho/\rho_0)$ would give a positive work for taking the molecule from within the liquid to without. It would be natural to try for a correspondence between $w = -kt \log (\rho/\rho_0)$ and some definition of latent heat of evaporation as representing the work done in carrying a molecule from the liquid to the gaseous phase. The following figures were prepared as an exercise in my course on the constitution of matter three years ago by Mr. Bailey Townsend:

	t(°C)	W	LATENT HEAT	INTERNAL L. H.
Water	0	370	595	565
Water	150	282	504	459
Benzene	0	62	94	87
Benzene	100	50	78	68
Benzene	250	25	21	14
Oxygen	−210	15	48	45
Oxygen	−140	8	33	30

Although in most cases w is less than either of these latent heats, we should not overlook the apparent fact that for benzene at 250°C the relation is reversed. It would also be possible to try to relate w to surface tension. And finally if we try to use the relation $w = -kt \log (\rho/\rho_0)$ in cases of

thermal dissociation of gases, e.g., $O + O = O_2 +$ heat, the densities or pressures would become simply expresssed in terms of concentrations and w would have to change sign at that degree of dissociation for which $\rho = \rho_0$ (density of O = density of O_2) without any obvious physico-chemical significance thereto.

With such considerations in mind I have been in the habit of saying that the Boltzmann equation $\rho = \rho_0 \exp(-w/kt)$ was no statistical or thermodynamic theorem of universal applicability and that in every case where it is used care should be taken to prove its availability or to specify the significance of w as work done or to emphasize the entirely tentative nature of its use. It was with this viewpoint that I spoke perhaps slightingly of the equation in my paper on metallic conduction in these PROCEEDINGS (**9**, p. 137, 1923; see also, E. H. Hall, *Ibid.*, **11**, p. 36, 1925) and I should not now be reverting to the matter had not persons in whose judgment I have confidence expressed the opinion that there is still enough confusion in the matter to make this note of service. It may be pointed out that if V be the potential energy of the external field and ϵ the intrinsic energy of the gas so that the complete heat function χ is $\epsilon + pv + V$, then we may show that $\Delta\chi = t\int(\partial p/\partial t)_v dv$ which in the case of van der Waal's equation leads to the relation per molecule.

$$(\Delta\chi) \text{ per molecule} = kt \log [(v - b)/(v_0 - b)] \quad \text{not} \quad kt \log (v/v_0);$$

and if we choose to take $w = (\Delta\chi)_{p.\,m.}$, we have a simple result in terms of the covolumes, though not simple in densities unless one wishes to measure density in terms of the volume unoccupied by molecules.

THE BAND SPECTRA OF CRYSTALS AND COMPLEX GASES

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The luminescent spectra of the uranyl salt crystals first discussed from the modern standpoint by Ehrenfest presents some radiation characteristics of a remarkable nature. These crystals at liquid-air temperatures as discovered by J. Becquerel and Onnes in 1903 have a large number of narrow bands occurring at definite intervals across the invisible spectrum. The experimental results of E. L. Nichols[1] and associates for the simplest cases of frequency arrangement we put in the following form. The spectrum consists of high frequency and low frequency branches, the two sets of branches having different spacing constants. For the high frequency

side (absorption) the interval is in round numbers, 700 units for a given series of bands

$$\nu = k + a_1 n_1 + \ldots . \qquad (1a)$$

and for the low frequency side (emission) the interval is 850,

$$\nu = k - a_2 n_2 + \ldots . \qquad (1b)$$

In addition there are several series of bands of slightly different constant interval. The complete spectrum involves an equation of the well-known band spectral type

$$\nu = k + (a_1' n_1' - a_1'' n_1'') + (a_2' n_2' - a_2'' n_2'') + \ldots + (b_1' n_1'^2 - b_1'' n_1''^2). + (b_2' n_2'^2 - b_2'' n_2''^2). \qquad (1c)$$

An important series of closely related investigations by Schaefer[2] has shown that the near infra-red bands of a large number of crystals are due to localized vibrations of definite atoms. Nichols found that in solutions the character of the spectrum was changed somewhat which we interpret as the effect of coupling upon the characteristic localized vibrations, due to the solvent molecules.

This evidence makes it unnecessary to consider the remark of Bohr, Kramers and Slater[3] that in macroscopic phenomena wherein the entire body takes part in the vibrations that the original conception of stationary states loses its meaning.

It has been demonstrated by Heurlinger, Kratzer,[4] Hulthen,[5] Runge and others that in many cases the different quantum transitions have a common initial or end state. Under these conditions the formula requires one less constant. Thus equation (1c) becomes on putting a_1'' equal to a constant, say zero

$$\nu = k + a_1' n_1' + + \ldots , , \quad \text{with spacing } a_1'. \qquad (1a')$$

(We are here concerned with the assumption that the transitions to or from certain stationary states are sufficiently characteristic to be identified On putting a_1' equal to a constant there results

$$\nu = k - a_1'' n_1'' + \ldots . , \quad \text{with spacing } a_1''. \qquad (1b')$$

The difference between the spacings in the two branches of the uranyl spectra is thus the difference between a_1' and a_1''.

That this effect is general and is independent of a constant interval can be seen by an example B^{11} 0.[6]

A related example is the investigation of T. Tanaka[7] showing the change in frequency interval of successive bands with change in atomic weight of the active metal in the spectra of solid solutions. The spectra seem to be of the same type as those of the uranyl salts dissolved in glass or fluxes. There are for each active metal in a solid solution equations of type 1a.

TABLE 1

n_1'	n_1''	ν	DIFF.	n_1'	n_1''	ν	DIFF.
5	0	48975.3		0	0	42889.8	
			1178.8				1860.9
4	0	47796.5		0	1	41019.9	
			1199.3				1837.5
3	0	46597.2		0	2	39182.4	
			1218.2				1814.0
2	0	45379.0		0	3	37368.4	
			1237.9				1790.4
1	0	44141.1		0	4	35578.0	
			1260.3				1764.7
0	0	42880.8		0	5	33813.3	

The longest series have as many successive bands as 39, over the whole range the interval being constant within the experimental error. This unusual combination of equally spaced bands indicating harmonic motion and large quantic transitions indicating non-harmonic motion may be satisfactorily solved in the following way. The motion in the initial stationary state contains higher harmonics leading to large transitions. The motion in the final state is simple harmonic containing insignificant quadratic terms and leads to equally spaced bands. This may be clearly illustrated by Henri's[8] data on the ultra-violet absorption of benzene.

TABLE 2

	$n_1 = 0$			$n_1 = 2$		
n_2	CALC.	OBS.	DIFF.	CALC.	OBS.	DIFF.
0	37494	37494		37168	37172	
			924			921
1	38415	38418		38089	38093	
			917			
2	39337	39335				
			924			
3	49258	49259		39932	39933	
			921			921
4	41179	41178		49854	40854	
			924			920
5	42101	42102		41775	41774	
			920			919
6	43022	43022		42696	42693	
			916			
7	43944	43938				

The calculated values are given by the formula

$$\nu = 37{,}494 + 921.4\, n_2 - 159\, n_1 - 2\, n_1^2 \qquad (2)$$

where n_1 refers to the initial state and n_2 refers to the final state.

In the data of Tanaka not only is there a continuous relation between the nuclear frequencies of all the metals but the absolute frequency range is so nearly the same for the strongest and best spectra that there must be

a common physical constituent in all the vibrating systems. The series electron orbit we consider as approximately fixed with respect to this constituent. Under these circumstances let the mass of the metal atom be M, the mass of the atoms vibrating with it p and let the mass of the oppositely vibrating constituent be q. Then so far as dependence on mass is concerned since $1/\mu = (1/[M + p]) + 1/q$ and $\nu \sim \sqrt{1/\mu}$ we see that the frequency should vary as $\sqrt{(1/[M + p]) + 1/q}$. For all values of p and q, however, this radical when plotted against M lies well above Tanaka's curve. Hence as would be expected the complications of the unknown ionic radii and crystalline coupling forces must be taken into consideration to get a closer approximation to the experimental curve.

Finally we may remark upon the meaning of the linear terms in equation 1c. Henri[9] and associates have found in the case of several organic vapors that the spectra must be expressed by the use of multiple linear terms. These spectra appear to be more general applications of simultaneous combination transitions used by Hettner in explaining infra-red oscillation bands for which no electron transition occurred. In line spectra this has been discussed by Wentzel, Saunders and Russel, Epstein and others.[10] In the gases studied by Hettner[11] the coefficients a were always of the same sign from which he concluded that the quantic numbers must all increase or all decrease. This rule does not hold when applied to the visible spectra of Henri, however.

The visible spectra gives a better insight into the phenomenon of multiple transitions because we may in some cases gain information about the nuclear quantum levels. Referring to the benzene data (equation 2) we see that one fundamental frequency is connected with the initial state corresponding in 1c to $a_1' n_1'$ while the other fundamental connected with the end state corresponds to $a_2'' n_2''$.

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THE FREQUENCY DISTRIBUTION OF SOME MEASURED PARALLAXES AND OF THE PARALLAXES THEMSELVES

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It is well known that the frequency functions of a set of quantities and of their measures are different;[1] if the error of measurement is comparable in magnitude to the quantity determined, they may be decidedly dissimilar. In the case of good photographic parallaxes the reported probable errors average about eight one-thousandths of a second of arc, making a dispersion of twelve. Moreover, as we have shown,[2] there appears to be a Lexian ratio of 1.25 here involved which would increase the true dispersion to $\sigma = 15$. This might well seriously disturb the frequency distribution of the large number of parallaxes which lie between $-0.''15$ and $+0.''050$.

To obtain a fairly comparable set of stars for discussion we restricted our list to a spread of one magnitude in apparent brightness and found 313 determinations[3] of parallax by the Allegheny, McCormick and Mt. Wilson observatories of stars between magnitudes 5 and 6. For some stars there were several determinations; it was decided to discuss the frequency distribution of the measures, and thus each measure was counted separately. To have combined by averaging, the plural measurements would have introduced a variety of different weights; to have chosen one of the measures and discarded the rest would have involved a more or less arbitrary choice.[4]

The frequency curve obtained is very skew, but looks like a logarithmic transform, and is, in fact, normal in $x = \log_{10}(\tilde{\omega} + 20)$ the parallax $\tilde{\omega}$ being taken in thousandths of a second of arc. Plotted on probability paper, the distribution nowhere departs from a straight line more than 2 or 3 per cent, which is about the magnitude of the errrors of sampling. The fit appears better than would be expected on chance (figure 1, line with points).

It may therefore be assumed that the frequency distribution of these parallax-measurements is:

$$f(x) = \frac{1}{\sqrt{2\pi}\tau} e^{-(x-x_0)^2/2\tau^2}, \quad x_0 = 1.625 \quad \tau = 0.275.$$

The method of passing from this to the frequency function of the parallaxes themselves, given by Whittaker and Robinson[1] is not very satisfactory on account of the slow convergence of the series

$$f(\tilde{\omega}) = e^{-D^2/2\sigma^2} \frac{\log_{10} e}{\sqrt{2\pi} + (\tilde{\omega} + 20)} e^{-[\log_{10}(\tilde{\omega}+20)-1.62]^2/2\tau^2}, \quad D = d/d\tilde{\omega}$$

The term D^4 cannot be neglected and the expressions become much involved. It is, however, possible to calculate the moments[5] of $f(x)$ on the basis of the variable $\tilde{\omega}$ and to express the moments of the true distribution (φ) in terms of them and of $\sigma = 15$.

Let $\log_{10} G = x_0 = 1.625$, $\log_{10} H = \tau^2/_2 . \log_{10} e = 0.087$,

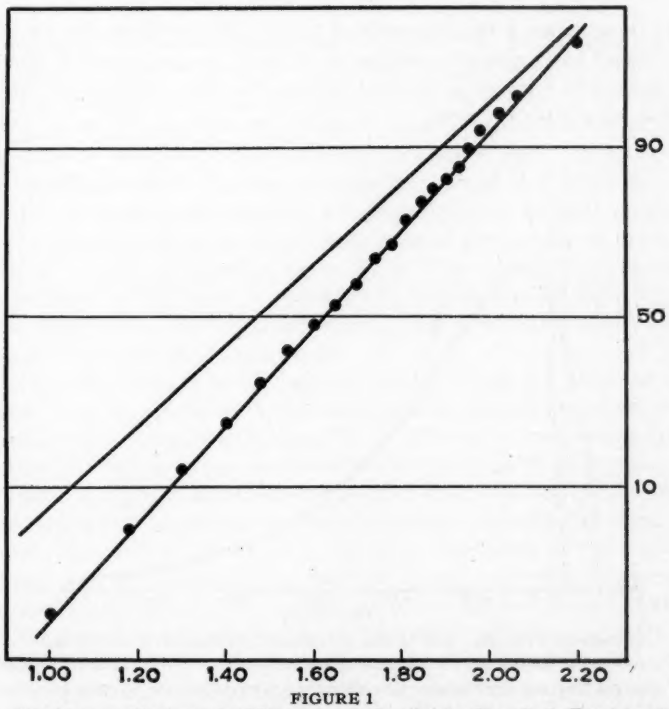

FIGURE 1

Plot on logarithmic-probability paper of the distribution of observed parallaxes (pointed line) with scale giving $\log_{10} (\tilde{\omega} + 20)$ and of the fitted logarithmic transform (above) with scale giving $\log_{10} (\tilde{\omega} + 6.3)$.

then G is the median of the distribution f and GH is its arithmetic mean measured from the origin $\tilde{\omega} = -20$. The moments about the mean are:

$$\mu_2 = G^2H^2(H^2 - 1), \qquad \mu_3 = G^3H^3(H^2 - 1)^2 (H^2 + 2)$$

$$\mu_4 = G^4H^4(H^2 - 1)^2 (H^8 + 2H^6 + 3H^4 - 3)$$

The observational errors distributed normally with $\sigma = 15$ do not disturb the mean and the third moment but change the second and fourth moment of the true frequency function into:[6]

$$\mu_2' = \mu_2 - \sigma^2, \qquad \mu_4' = \mu_4 - 6\sigma^2\mu_2 + 3\sigma^4.$$

We have $\log_{10} GH = 1.712$, $GH = 51.5$, and the mean of $\tilde{\omega}$ is 31.5. Furthermore $\mu_2 = 1308$, $\mu_3 = 116{,}000$, $\mu_4 = 26{,}400{,}000$; and if the value of σ be 15 we have $\mu_2' = 1083$, $\mu_4' = 24{,}600{,}000$. This gives for the new curve the Pearsonian constants:

$$\beta_1 = 10.6 \qquad \beta_1^{1/2} = 3.26, \qquad \beta_2 = 21.0 \quad k_1 = 4, \qquad k_2 = 7.4.$$

With these values a Pearson curve of type VI may be fitted, but the origin thus found has apparently nothing to do with the astronomical problem, the numerical formula is involved, and the solution is generally unsatisfactory, as is often the case.

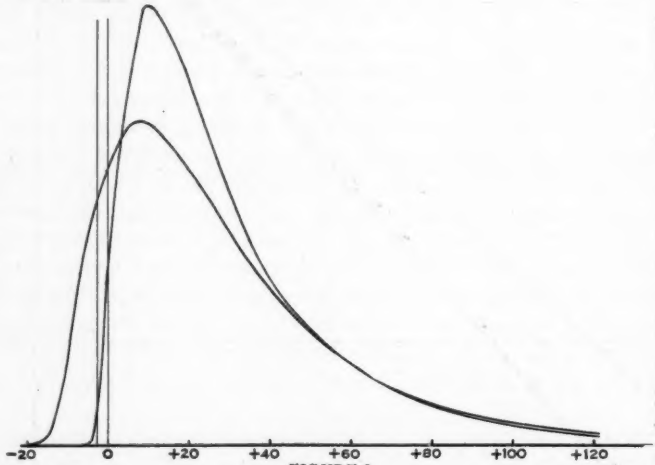

FIGURE 2

Graphs on a parallax base of the frequency distributions given in figure 1. The vertical line to the left of the origin is located at $\tilde{\omega} = -2.7$ and represents the true origin of parallaxes when allowance is made for the average parallax of the comparison stars. The lower curve gives the distributions of observations and the higher curve of the parallaxes themselves.

There is, however, available another method which may be followed. The transformed curve must in this case be a good deal like a logarithmic transform with the origin drawn in from $\tilde{\omega} = -20$ toward $\tilde{\omega} = 0$. As a first approximation we may determine that logarithmic transform which has its first three moments identical with the above computed for the transformed true frequency curve.

Let the new origin be at a, on the scale of $\tilde{\omega} + 20$. Then

$$GH = 51.5 = G'H' + a; \qquad (G'H')^2 (H'^2 - 1) = u_2' = u_2 - \sigma^2 = 1083$$

$$\mu_3 = \mu_3' = 116{,}000 = (G'H')^3 (H'^2 - 1)^2 (H'^2 + 2).$$

These equations may be solved for G'; H'; and a, to give

$$a = 13.7, \quad G' = 28.7, \quad H' = 1.32, \quad H'^2 = 1.75, \text{ and}$$

$$f(\tilde{\omega}) = \frac{\log_{10} e}{\sqrt{2\pi}\,\rho(\tilde{\omega} + 6.3)}\, e^{-[\log_{10}(\tilde{\omega}+6.3)-1.46]^2/2\rho^2}, \quad \rho = 0.324.$$

The observed median or geometric mean is 41.7 − 20 = 21.7; the median or geometric mean ex observational errors is 28.7 − 6.3 = 22.4. The observed mode is $G/H^2 = 27.6$ measured from the origin $\tilde{\omega} = -20$, or at parallax $\tilde{\omega} = 7.6$; the corrected mode is at $G'/H'^2 = 16.4$, or at $\tilde{\omega} = 16.4 - 6.3 = 10.1$. There are no parallaxes under —6.3; there are only seven parallaxes out of 311 that are negative, instead of 41 observed. As there is a systematic correction[2] of 2.7 which should be added to the parallaxes to reduce them to absolute values some of the parallaxes might be negative. In this distribution log (—2.7 + 6.3) = log 3.6 = 0.56 which departs from 1.46 by $0.90 = 2.8\rho$ and occurs only once in 350 times, or once in our series instead of 25 times as before. The new distribution is therefore a great improvement.

The fourth moments of the new and old curves do not check as they should. But the maximum contribution to these moments on either curve lies out in the region of parallaxes 200 to 250 where observations are so few that no confidence can be placed in the values of the moments. Indeed the maximum contribution to the third moments, though not nearly so far out, is still far enough to render somewhat doubtful the identification of $M_3 = M_3' = 116{,}000$ as a device to determine the new curve. Consequently the location of the mode on the new curve and the minutiae of its configuration cannot be regarded as well established; but we believe that its general characteristics are as portrayed and that it gives a far better picture of the frequency distribution of these parallaxes than does the original curve. The example serves to point out how seriously two such frequency distributions may differ (fig. 2).

[1] Whittaker and Robinson, *Calculus of Observations*, Art. 105, p. 206.

[2] These PROCEEDINGS, **10**, 129 (1924).

[3] Two parallaxes, at —25 and —31, were rejected.

[4] On the whole, the procedure followed seems as good as any, though it makes the frequency distribution of the parallax-measurements slightly different from that in which no star was represented by more than one parallax. The effect is most pronounced in the part of the distribution determined by the larger parallaxes, which is the part of less interest in the present discussion.

[5] See, e.g., Arne Fisher, *Mathematical Theory of Probabilities*, 2nd Ed., Chap. XVI, p. 237.

[6] If a frequency curve $\varphi(x)$ has each of its elements $\varphi\,dx$ subject to dispersion by a normal error function, the resulting observed frequency function $f(\xi)$ is

$$f(\xi) = \int_x \frac{\varphi(x)dx}{\sqrt{2\pi\sigma}}\, e^{-(\xi - x)^2 : 2\sigma^2}.$$

Then $$\int_\xi \xi^k f(\xi)d\xi = \int_\xi \int_x [(\xi - x) + x]^k \frac{\varphi(x)}{\sqrt{2\pi}\sigma} e^{-(\xi-x)^2 : 2\sigma^2} d\xi\, dx.$$

As $d(\xi - x) = d\xi$, the integration in ξ may be performed as

$$\int_\xi \xi^k f(\xi)d\xi = \int_x (M_k + kM_{k-1}\, x + \tfrac{1}{2} k(k-1)M_{k-2}\, x^2 + \ldots + x^k)\, \varphi(x)dx$$

where M_k are the moments of order k for the normal frequency function and are simply expressible in terms of σ (the moments of odd order vanish). Next an integration with respect to x gives

$$\nu_k = \int_\xi \xi^k f(\xi)d\xi = M_k + kM_{k-1}\, \nu'_1 + \tfrac{1}{2} k(k-1)M_{k-2}\, \nu'_2 + \ldots + \nu'_k$$

where ν are the moments about any origin the observed frequency function f and ν' those about the same origin for the true frequency function φ.

THE EFFECT OF VARYING MASS ON A BINARY SYSTEM

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1. In the *Monthly Notices* for 1924, November, Dr. Jeans considers the effect on a variable mass M of a central acceleration M/r^2. He writes down the equations in polar coördinates, namely,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{M}{r^2}, \qquad r^2\left(\frac{d\theta}{dt}\right) = \text{const.}, \tag{1.1}$$

and the equation deducible from these, namely,

$$\frac{d}{dt}\left\{\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2} r^2\left(\frac{d\theta}{dt}\right)^2 - \frac{M}{r}\right\} = -\frac{1}{r}\frac{dM}{dt}. \tag{1.2}$$

He puts $-M/2a$ for the terms in brackets where $2a$ is the major axis in an elliptic orbit, so as to obtain

$$\frac{d}{dt}\left(\frac{M}{2a}\right) = \frac{1}{r}\frac{dM}{dt}. \tag{1.3}$$

Then, pointing out that the mean value of $1/r$ is $1/a$, he deduces

$$Ma = \text{const.} \tag{1.4}$$

2. Suppose, however, we should write (1.2) in the form

$$\frac{d}{dt}\left\{\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2} r^2\left(\frac{d\theta}{dt}\right)^2 - \frac{M_0}{r}\right\} = (M - M_0)\frac{d}{dt}\left(\frac{1}{r}\right) \tag{2.5}$$

and put $-M_0/2a$ for the portion in parentheses in the left hand member, we should get

$$-\frac{d}{dt}\left(\frac{M_0}{2a}\right) = (M - M_0)\frac{d}{dt}\left(\frac{1}{r}\right), \qquad (2.6)$$

where M_0 is a constant. If we adopt the same argument as before, namely put $1/a$ for the mean value of $1/r$ in (2.6) we should obtain

$$a = \text{const.} \qquad (2.7)$$

The difference between (2.4) and (2.7) might be explained as due solely to the different definitions of a in the two cases. But then we should be in a difficulty as to which was correct. As a matter of fact, the definition used in (2.6) is that ordinarily adopted in the method of the variation of arbitrary constants which the former is not, on account of the variability of M. The real source of the error, however, arises from the assumption that the disturbed mean value of $1/a$ gives the disturbed mean value of $1/r$. The fact is that there is a secular term which arises from the periodic part of $1/r$. If $M - M_0$ be a small quantity of the first order, the usual method for the variations of the elements shows that the secular change of $1/a$ is of the second order, as indeed appears from (2.6) which is the equation for $1/a$, and that the secular part arising from the periodic part of $1/r$ is of the first order. In symbols, if e be the eccentricity and l the mean anomaly, $\delta(e \cos l)$ produces a secular part independent of the angle l.

3. Instead of proceeding to calculate this additional part, we can obtain a solution more simply in the following way. If we transform to the variables $u, = 1/r$, and θ by making use of the equation $r^2 d\theta/dt = h = $ a constant, we obtain

$$d^2u/d\theta^2 + u = M/h^2 \qquad (3.1)$$

the solution of which may be written

$$u = 1/r = M'/h^2 + A\cos(\theta - \tilde{\omega}) \qquad (3.2)$$

where $A, \tilde{\omega}$ are constants and M' is a particular solution of

$$d^2M'/d\theta^2 + M' = M. \qquad (3.3)$$

The cases of interest are those in which the variation of $\log M$ is very small compared with the variation of θ during a revolution of the system. I shall show in a moment what is perhaps obvious, namely, that it makes little difference whether, in this case, we regard M as a function of t or as a function of θ. Adopting the latter, we can express the slow variation of M by putting

$$M = f(k\theta),$$

where k is very small, the derivatives of f with respect to $k\theta$ remaining finite. In this case, we can find the required particular integral by putting $D = d/d\theta$, and using the symbolic development

$$M' = (1 + D^2)^{-1}(k\theta) = M - k^2f'' + k^4f^{iv} + \ldots \qquad (3.4)$$

a first approximation to which is given by $M' = M$.

Since M' varies very slowly, we can compare the orbit (3.2) at any time with the ellipse

$$\frac{1}{r} = \frac{1 + e\cos(\theta - \tilde{\omega})}{a(1 - e^2)}. \tag{3.5}$$

If we do so, we obtain

$$e = \frac{Ah^2}{M'}, \qquad \frac{1}{a} = \frac{M'}{h^2}\left(1 - \frac{A^2h^4}{M'^2}\right), \tag{3.6}$$

which involves the equation $M'a\,(1 - e^2) = \text{const.}$

As to the point laid aside for a moment, let us suppose $M = \phi\,(knt)$. We then have $nt = \theta - E$ where E is a periodic function of t, having a period approximately that of the revolution of the system. The equation (3.4) shows that M' still differs from M by terms having the factor k^2 so that the argument of the previous paragraph is not altered.

The ordinary procedure of approximation may be adopted if we wish to take k^2 into account. But there is no need to do so. As long as we know that the rates of change of a, e retain the same sign throughout the whole of the period under discussion, the main object in view here will be attained.

4. The definition of a in this method is not that adopted in the method for the variation of the elements. In the latter, the elements are so varied that the velocities and coördinates retain the same form, when expressed in terms of t and the elements, as they have in undisturbed motion. If, however, the coördinates are finally expressed in terms of the time and real constants we should obtain a solution which would be the same as (3.2). The mean value of u through a revolution in this equation is the mean value of M'/h^2 since $A,\tilde{\omega}$ are constants. The comparison to an ellipse is not necessary here: it is simply a convenient way of picturing the orbit.

5. The results obtained in §3, can be stated quite briefly. Both "*mean distance*" a and "*eccentricity*" e *increase as* M *decreases*, the relation between these rates being expressed by the equation $a/e - ae = \text{const.}$, and the meanings attached to a, e being defined in the same section. The areal velocity is constant.

6. In the paper referred to, Dr. Jeans considers the effect of a decrease of mass on a binary system. For this purpose, he assumes that

$$\text{mass} \times \text{acceleration} = \text{force}. \tag{Hyp. 1}$$

The usual form of the second law of motion is, however,

$$\text{rate of change of momentum} = \text{force}. \tag{Hyp. 2}$$

Another form which suggests itself is

$$d^2/dt^2\,(\lambda x) = \lambda \times \text{force along the } x\text{-axis}, \tag{Hyp. 3}$$

where λ is a function of the masses only.

These reduce to the same equation if the masses are constant, but there appears to be no information as to which should be chosen if mass is lost in any other way than that given by the classical laws of dynamics.

There is the further question as to the use of a frame of reference. I shall only examine here what the three hypotheses produce when referred to a Newtonian frame.

Let (x_1, y_1), (x_2, y_2) be the coördinates of the masses m_1, m_2 referred to a Newtonian frame, and r their distance apart. Put

$$x = x_2 - x_1, \qquad (m_1 + m_2)\bar{x} = m_1x_1 + m_2x_2, \tag{6.1}$$

with similar equations for y_1, $\bar{y}$. Then $(\bar{x}, \bar{y})$ are the coördinates of the center of mass, while (x, y) are those of m_2 relative to m_1. This is a more convenient method than that of referring the motion of each body to the center of mass.

7. Under the first hypothesis, the equations of motion are

$$m_1 \frac{d^2x_1}{dt^2} = -m_1m_2\frac{x_1 - x_2}{r^3}, \qquad m_1 \frac{d^2x_2}{dt^2} = -m_1m_2\frac{x_2 - x_1}{r^3}, \tag{7.1}$$

with similar equations for y_1, y_2. By the use of (6.1) these give

$$\frac{d^2x}{dt^2} = -(m_1 + m_2)\frac{x}{r^3}, \qquad \frac{d^2y}{dt^2} = -(m_1 + m_2)\frac{y}{r^3}. \tag{7.2}$$

Here (§3) $M = m_1 + m_2$, so that M diminishes when the total mass of the system diminishes, and the result in §5 is directly applicable.

For the motion of the center of mass we have, from (7.1),

$$m_1\, d^2x_1/dt^2 + m_2\, d^2x_2/dt^2 = 0,$$

which becomes

$$\frac{d^2\bar{x}}{dt^2} = 2\frac{dx}{dt}\frac{dp}{dt} + x\frac{d^2p}{dt^2}, \qquad 2p = \frac{m_1 - m_2}{m_1 + m_2}. \tag{7.3}$$

If d^2p/dt^2 is small compared with dp/dt (§3), we can integrate and obtain

$$d\bar{x}/dt = 2x\, dp/dt + \text{const.}, \tag{7.4}$$

so that the velocity of the center of mass of the system increases as m_2/m_1 decreases. We can always assume that $m_2 < m_1$ and it is probable that their rates of change follow the same law. Thus, unless $m_1 = m_2$ throughout, or $m_2 = 0$, which cases give constant velocity, *the velocity of the center of mass increases.*

The angular momentum obviously decreases, the areal velocity remaining constant.

8. The second hypothesis gives

$$\frac{d}{dt}\left(m_1\frac{dx_1}{dt}\right) = -m_1m_2\frac{x_1 - x_2}{r^3},$$

with corresponding equations for x_2, y_1, y_2. This system possesses the integrals,

$$m_1\,dx_1/dt + m_2\,dx_2/dt = a, \qquad m_1\,dy_1/dt + m_2\,dy_2/dt = b,$$
$$m_1(x_1dy_1/dt - y_1\,dx_1/dt) + m_2(x_2\,dy_2/dt - y_2\,dx_2/dt) = H,$$

where a, b, H are constants.

If we can choose our axes of reference that $a = 0 = b$, the equation for x becomes

$$\frac{d}{dt}\left(\frac{m_1\,m_2}{m_1 + m_2}\,\frac{dx}{dt}\right) = -\,m_1\,m_2\,\frac{x}{r^3}$$

or, if we put $m_1m_2dt' = (m_1 + m_2)dt$,

$$\frac{d^2x}{dt'^2} = -\,\frac{m_1^2\,m_2^2}{m_1 + m_2}\,\frac{x}{r^3}.$$

The change from t to t' obviously makes no difference to the equation to the orbit. Here (§3)

$$\frac{1}{M} = \frac{1}{m_1\,m_2^2} + \frac{1}{m_2\,m_1^2},$$

so that M decreases with m_1, m_2 and the result of §5 is again applicable.

The angular momentum here remains constant while the areal velocity increases.

The motion of the center of mass is derived from

$$\frac{d\bar{x}}{dt} = x\,\frac{dp}{dt}, \qquad p = \frac{1}{2}\,\frac{m_1 - m_2}{m_1 + m_2}$$

so that the result is like that found under the first hypothesis.

9. For the third hypothesis we so choose λ that the center of mass shall have no acceleration, and that the equations shall reduce to the usual form when the masses are constant. The equation for x_1 will then be

$$\frac{d^2}{dt^2}\left(\frac{m_1\,x_1}{m_1 + m_2}\right) = -\,\frac{m_1\,m_2}{m_1 + m_2}\,\frac{x_1 - x_2}{r^3}.$$

If the origin be taken at the center of mass the equation for x will be

$$\frac{d^2}{dt^2}\left\{\frac{m_1\,m_2}{(m_1 + m_2)^2}\,x\right\} = -\,\frac{m_1\,m_2}{m_1 + m_2}\,\frac{x}{r^3}.$$

If we put

$$m_1\,m_2\,x = (m_1 + m_2)^2x',$$

with the same proportional changes for y, r, the equations reduce to the form (1.1) with

$$M = \frac{m_1^3\,m_2^3}{(m_1 + m_2)^3},$$

and the orbit to

$$\frac{1}{r} = \frac{m_1 m_2}{(m_1 + m_2)^2} \left\{ \frac{M'}{h^2} + A \cos (\theta - \tilde{\omega}) \right\}.$$

The eccentricity e is inversely proportional to M', while

$$\frac{1}{a} = \frac{(m_1 + m_2)^2}{m_1 m_2} \cdot \frac{M'}{h^2} \left\{ 1 - \frac{A^2 h^4}{M'^2} \right\}.$$

The rates of change of a, e here depend on the values of m_1, m_2 and consequently on the rates of change.

We have

$$\frac{dM}{M} = \frac{dm_1 (3m_2 - 2m_1)}{m_1(m_1 + m_2)} + \frac{dm_2 (3m_1 - 2m_2)}{m_2(m_1 + m_2)}.$$

We can always assume that $m_2 > m_1$. If $2m_2 < 3m_1$ for all time, M (or M') diminishes with m_1, m_2 and the same is true of $(m_1 + m_2)^2 M/m_1 m_2$.

If $m_2 < 2m_1$, and the eccentricity not too large the distance diminishes with m_1, m_2, an interesting case in view of the determinations of the relative masses in stellar binary systems.

9. Of the three hypotheses, there is little to distinguish between the first and second. Both of them give increases of distance and eccentricity and velocity of the center of mass, with diminishing masses, the chief distinction being that in the first the angular momentum increases while in the second it remains constant. This simultaneous change of distance and eccentricity has a well-known application to binary stellar systems.

The choice of the second of these hypotheses against the others has a further reason in the fact that with it, the velocity of a star as well as that of the center of mass of a system increases. This provides for the increase of velocity with age on Russell's scheme of evolution. To provide for the later decrease, it may be noted that the velocity of a single star relative to the center of mass of the whole stellar system will decrease. We thus have two opposing changes in the velocity of a star proceeding at different rates and it may be that these happen to cancel one another on the average when type B or A is reached.

ERRATA

Correction to the paper "Projective Normal Coördinates for the Geometry of Paths," by O. Veblen and J. M. Thomas, these PROCEEDINGS, **11**, pp. 204–7. The first formula on page 207 should read

$$Q^i_{jkl} = E^i_{jkl} - \tfrac{1}{3} S \left[E^i_{jkl} + \frac{2\delta^i_j}{n-1} (E^\alpha_{kl\alpha} - E^\alpha_{akl}) \right] + \frac{1}{n-1} [\delta^i_j (E^\alpha_{kl\alpha} - E^\alpha_{akl}) + \delta^i_k (E^\alpha_{jl\alpha} - E^\alpha_{ajl})].$$

There should also be a minus sign preceding a_j in formula (2.10).—J. M. THOMAS